Lecture 3

Monday, September 9, 2024 4:23 AM

* Supremum and infimum

$A \subset \mathbb{R}$ is a nonempty subset.

$\sup A =$ the smallest upper bound of A

$\inf A =$ the greatest lower bound of A

To show that $\sup A = m$, we need to show that

$$\begin{cases} m \text{ is an upper bound of } A \\ \text{If } M \text{ is an upper bound of } A \text{ then } M \geq m. \end{cases}$$

* A very important application of the lowest upper bound property is the Archimedean property:

- If $x > 0$ then there exists $n \in \mathbb{N}$ such that $nx > 1$.
- If $x > y$ then there exists $r \in \mathbb{Q}$ such that $x > r > y$.

Proof of (a) by contraction: Suppose that $nx \leq 1$ for every $n \in \mathbb{N}$.

Then $n \leq \frac{1}{x}$ for every $n \in \mathbb{N}$

That means $\frac{1}{x}$ is an upper bound of $\mathbb{N}$. Thus, $\mathbb{N}$ has a supremum, called M.

Because $M - 1 < M$, it is not an upper bound of $\mathbb{N}$.

There exists $m \in \mathbb{N}$ such that $m > M - 1$. Then $\underbrace{m+1}_{\in \mathbb{N}} > M$. Then M

is not an upper bound of $\mathbb{N}$. This is a contradiction

$\underline{\underline{Ex}}$ Let $A=[1,2)$. Find $\sup A$ and $\inf A$

$\underline{\underline{Ex}}$ Let $A=\left\{\frac{n}{n+1},\ n=1,2,3,\ldots\right\}$

Find $\sup A$ and $\inf A$.

$\underline{\underline{Ex}}$ Let f, g be bounded functions from $[0,1]$ to $\mathbb{R}$.

$A = \sup\{f(x) \mid x \in [0,1]\}$

$B = \sup\{g(x) \mid x \in [0,1]\}$

$C = \sup\{f(x)+g(x) \mid x \in [0,1]\}$

Show that $\sup C \leq \sup A + \sup B$

$\inf C \geq \inf A + \inf B$

Give an example where the equality doesn't hold.