Lecture 4

Wednesday, September 11, 2024 9:16 AM

I should have in the axioms of real numbers that:

- $0 \neq 1$
- Every subset that is nonempty and bounded from above has a supremum.

* How do you show that $1 > 0$?

Suppose that $1 < 0$.

Then $0 < -1$. Because $-1 > 0$ and $-1 > 0$, we have $(-1)(-1) > 0$. $(*)$

Claim: $(-1)(-1) = 1$

How do I show this? $(-1)(-1) + (-1) = (-1)(-1+1) = (-1)0 = 0$

So, $(-1)(-1)$ is the additive inverse of -1. Therefore $(-1)(-1) = 1$.

$(*)$ means $1 > 0$. This is a contradiction.

Absolute value

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

A very important property of the absolute value is the triangle inequality.

$$|x+y| \leq |x| + |y|$$

Ex Show that for any $n \geq 1$, and for any $x_1, x_2, \ldots, x_n \in \mathbb{R}$,

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$$

Proof by induction on n.

Ex Show that $|xy| = |x||y|$.

Ex Show that for any $n \geq 1$, and for any $x_1, x_2, \dots, x_n \in \mathbb{R}$,

$$|x_1 x_2 \cdots x_n| = |x_1||x_2| \cdots |x_n|$$

Ex Show that for any $x \in \mathbb{R} \setminus \{0\}$,

$$|x^{-1}| = \frac{1}{|x|}$$

Bounded functions

$f : D \to \mathbb{R}$ is bounded from above if there exists $M \in \mathbb{R}$ such that

$$f(x) \leq M$$

In other words, f is bounded from above if the range of f is bounded from above.

Ex Find a bound of $f(x) = \dfrac{x^3 - x^2}{x+1}$ for $x \in [2,5]$.

$$|f(x)| = \left| \frac{x^3 - x^2}{x+1} \right| = \frac{|x^3 - x^2|}{|x+1|} \leq \frac{|x^3| + |-x^2|}{|x+1|} = \frac{|x|^3 + |x|^2}{|x+1|}$$

$$\leq \frac{|x|^3 + |x|^2}{|x| - 1} \leq \frac{5^3 + 5^2}{2 - 1} = 150$$