Lecture 7

Wednesday, September 18, 2024 9:04 AM

Subsequence of (x_n): $x_1, x_2, x_3, x_4, \ldots$

is a sequence $x_{n_1}, x_{n_2}, x_{n_3}, x_{n_4}, \ldots$ where $n_1 < n_2 < n_3 < \ldots$

For example, $x_2, x_4, x_6, x_8, \ldots$ is a subsequence, but $x_1, x_2, x_1, x_2, x_1, x_2 \ldots$

is not a subsequence.

Property: $\lim x_n = L$ if and only if $\lim_{k\to\infty} x_{n_k} = L$ for any

subsequence (x_{n_k}) of (x_n).

Ex Show that the sequence $x_n = \sin(n)$ doesn't converge.

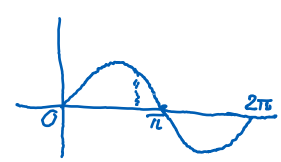
Suppose by contradiction that $\lim x_n = a$.

For each $k \in \mathbb{N}$, let n_k be the unique integer such that

$$2k\pi \le n_k < 2k\pi + 1$$

Notice that $n_1 < n_2 < n_3 < \ldots$ So, we have a subsequence (x_{n_k}).

$$\underbrace{\sin(2k\pi)}_{0} \le x_{n_k} < \underbrace{\sin(2k\pi+1)}_{\sin(1)}$$



Thus, $a = \lim_{k\to\infty} x_k \geq 0$

For each $k \in \mathbb{N}$, let m_k be the unique integer such that

$$2k\pi + 2 \le m_k < 2k\pi + 3$$

$\rightsquigarrow \sin 2 \le x_{m_k} \le \sin 3 < 0 \rightsquigarrow a < 0 \rightsquigarrow$ Contradiction

* Important properties of sequence:

If (a_n) is increasing and bounded from above then it converges.

If (a_n) " decreasing " " " below " " .

$a = \sup\{a_n : n \in \mathbb{N}\}$

We'll show that $\lim_{n\to\infty} a_n = a$.

Let $\varepsilon > 0$. We want to find N such that $|a_n - a| < \varepsilon$ for all $n > N$.

$$|a_n - a| = a - a_n$$

We want $\underbrace{a - a_n < \varepsilon}_{\text{equiv. to } a_n > a-\varepsilon}$ for $n > N$

Because a is the lowest upper bound of $\{a_n \mid n \in \mathbb{N}\}$, $a-\varepsilon$ is not an upper bound. Thus, there exists a_k such that $a_k > a - \varepsilon$. Since (a_n) is increasing,

$$a - \varepsilon < a_k \le a_{k+1} \le a_{k+2} \le \dots$$

Choose $N = k$ and we are done.

<u>Ex</u> $\quad a_{n+1} = \dfrac{4a_n}{3a_n + 3}$, $a_1 = 1$

Show that (a_n) is a decreasing sequence and $a_n > \frac{1}{3}$ for all n.

Find the limit of (a_n).