Lecture 8

Monday, September 23, 2024 4:19 PM

Superior and inferior limits of a sequence.

Note that a sequence has uncountably infinitely many subsequences.

__Theorem__: There exists a subsequence that has a maximum limit (compared to all other subsequences). This limit is called the superior limit of (a_n), denoted by $\limsup_{n\to\infty} a_n$ or $\overline{\lim_{n\to\infty}} a_n$.

The inferior limit is denoted by $\liminf_{n\to\infty} a_n$ or $\underline{\lim_{n\to\infty}} a_n$.

Ex $\quad a_n = \dfrac{n^2}{2n^2-7}$

$\lim a_n = \frac{1}{2}$

Any subsequence of (a_n) must converge to $\frac{1}{2}$. Thus,

$$\limsup a_n = \liminf a_n = \frac{1}{2}$$

__Remark__:

if $\lim a_n$ exists (possibly $\pm\infty$) then $\lim a_n = \limsup a_n = \liminf a_n$.

Ex $\quad a_n = \sin(n)$

$\limsup a_n = 1$

$\liminf a_n = 1$

Take a look at the paper

https://maa.tandfonline.com/doi/epdf/10.1080/0020739830140417?needAccess=true

for a more general result.

If f is period with an irrational period then $\limsup f(n) = \sup f$

and $\liminf f = \inf f$.

Ex $\quad a_n = \frac{2}{5} + \frac{n+50}{n} \sin(3n)$

Visualize this sequence by ploting the points (n, a_n). To enhance the visual effect, plot the points $(\frac{1}{n}, a_n)$.

On Mathematica:

$$a[n\_] := \frac{2}{5} + \frac{n+50}{n} \, Sin[3n]$$

$$seq = Table[\{\tfrac{1}{n}, a[n]\}, \{n, 1, 10000\}]$$

$$ListPlot[seq]$$

Alternative definition of limsup, liminf:

$$\alpha_n = \inf\{a_k : k \geq n\}, \quad \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \cdots$$

$$\beta_n = \sup\{a_k : k \geq n\}, \quad \beta_1 \geq \beta_2 \geq \beta_3 \geq \cdots$$

(α_n) and (β_n) each has a limit.

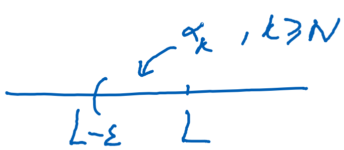
$$\liminf a_n \stackrel{\text{def}}{=} \lim \alpha_n$$

$$\limsup a_n \stackrel{\text{def}}{=} \lim \beta_n$$

Theorem

Every bounded sequence has a convergent subsequence.

Proof ....



We show that there exists a subsequence (a_{n_k}) such that $a_{n_k} \to \limsup a_n$