Lecture 9

Wednesday, September 25, 2024 9:20 AM

Recall two equivalent definitions of limsup and liminf:

① $\limsup a_n = \max\{\lim_{k\to\infty} a_{n_k} \mid (a_{n_k}) \text{ has a limit (possibly } \pm\infty)\}$

$\liminf a_n = \min\{\ \ldots\ldots\ \}$

② $\limsup a_n = \lim_{n\to\infty} \sup\{a_k \mid k \geq n\}$

$\liminf a_n = \lim_{n\to\infty} \inf\{a_k \mid k \geq n\}$

Note that $\limsup a_n$ and $\liminf a_n$ always exist (possibly equal to $\pm\infty$) even if $\lim a_n$ doesn't exist.

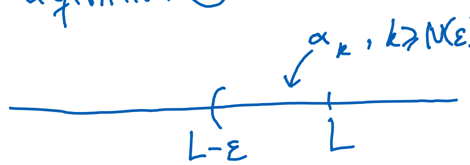
<u>Bolzano-Weierstrass theorem</u>: Every bounded sequence has a convergent subsequence.

* Proof using definition ① of limsup, liminf:

Let (x_{n_k}) be the subsequence such that $\lim x_{n_k} = \limsup x_n$.

We only need to explain why $\limsup x_n < \infty$. Suppose by contradiction that $\limsup x_n = \infty$. Then $\lim x_{n_k} = \infty$. Then for any $M > 0$, there exists $N(M) \in \mathbb{N}$ such that $x_{N(M)} > M$. Thus, (x_n) is unbounded. This is a contradiction.

* Proof using definition ②:



For each $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that

$$L - \varepsilon < \alpha_k \leq L \quad \forall k \geq N(\varepsilon)$$

For each $k \geq N(\varepsilon)$, there exists $j_{k,\varepsilon} \geq k \geq N(\varepsilon)$ such that

$$\alpha_k \leq x_{j_{k,\varepsilon}} < \alpha_k + \varepsilon$$

Let $n_1 = j_{N(1),1}$ and

$$n_{m+1} = j_{k,\frac{1}{m+1}} \text{ where } k = \max\{N(\tfrac{1}{m}), n_m + 1\}$$

Then $n_{m+1} \geq k \geq n_m + 1 > n_m$ and

$$x_{n_m} = x_{j_{k,\frac{1}{m}}} < \alpha_k + \tfrac{1}{m} \leq L + \tfrac{1}{m}$$

$$x_{n_m} = x_{j_{k,\frac{1}{m}}} \geq \alpha_k > L - \tfrac{1}{m}$$

Therefore, (x_{n_m}) is a subsequence with $\lim x_{n_m} = L$.

<u>Cauchy sequence</u>

* <u>Def</u> : (a_n) is called a Cauchy sequence if

$$\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N} : \ |a_n - a_m| < \varepsilon \quad \forall m, n > N$$

* <u>Theorem:</u>

A sequence is a Cauchy sequence if and only if it converges to a finite limit.

<u>Ex</u> Consider the sequence $a_1, a_2 \in \mathbb{R}$, $a_{n+1} = \dfrac{a_n + a_{n-1}}{2}$.

Show that (a_n) converges and find $\lim a_n$.