Week 2

Wednesday, September 10, 2025 11:26 AM

Goals:

- Basic concepts and properties of sequences.
- Technique of investigating the convergence of recursive sequences of the form $x_{n+1} = f(x_n)$.

Definitions:

- A sequence $\{x_n\}$ is called **increasing** (decreasing) if for every n we have $x_{n+1} \geq x_n$ ($x_{n+1} \leq x_n$). An increasing or decreasing sequence is collectively called a **monotone sequence**.
- A sequence $\{x_n\}$ is called **bounded above** if there exists a real number M such that for every n, $x_n \leq M$.
- A sequence $\{x_n\}$ is called **bounded below** if there exists a real number m such that for every n, $x_n \geq m$.
- A sequence that is both bounded above and bounded below is called a **bounded sequence**.
- A sequence $\{x_n\}$ is called **periodic with period k** if for every $n \in \mathbb{N}$, $x_{n+k} = x_n$.
  A sequence with period 1 is called a **constant sequence**.

Theorem 1. (Sum, difference, product, quotient of convergent sequences)

If $\{x_n\}$, $\{y_n\}$ are convergent sequences with limits a, b respectively, then the sequences $\{x_n \pm y_n\}$, $\{x_n y_n\}$, $\{x_n / y_n\}$ (provided $b \neq 0$) are also convergent with limits $a \pm b$, ab, a/b.

Theorem 2. (Passing limits through inequalities)

Suppose $\{x_n\}$ has finite limit ℓ. If there exists $N_0 \in \mathbb{N}$ such that for all $n > N_0$, $a \leq x_n \leq b$, then $a \leq \ell \leq b$.

Theorem 3. (Squeeze Theorem)

Let $\{x_n\}, \{y_n\}, \{z_n\}$ be sequences where $\{x_n\}$ and $\{z_n\}$ have the same finite limit L, and there exists $N_0 \in \mathbb{N}$ such that for all $n > N_0$, $x_n \leq y_n \leq z_n$. Then $\{y_n\}$ also has limit L.

Theorem 4. If a sequence $\{x_n\}$ converges, then $\{x_n\}$ is bounded.

Theorem 5. (Monotone Sequence Theorem)

An increasing sequence bounded above or a decreasing sequence bounded below is convergent.
In short, a monotone and bounded sequence converges.

Theorem 6. (Bolzano–Weierstrass Theorem)

Every bounded sequence has a convergent subsequence.

Theorem 7.

Let I be a closed interval of $\mathbb{R}$ and let $f : I \rightarrow I$. Consider the sequence $\{x_n\}$ defined by

$x_0 = a \in I, x_{n+1} = f(x_n)$ for $n = 0,1,2,\ldots$

1. If f is increasing on I, then $\{x_n\}$ is monotone. The sequence is increasing or decreasing depending on the position of x_0 relative to x_1.
2. If f is decreasing on I, then the subsequences $\{x_{2k}\}$ and $\{x_{2k+1}\}$ are monotone (and in opposite directions).
3. Suppose f is continuous on I. If $lim\, x_n = L$, then $L \in I$ and passing to the limit in the recurrence relation $x_{n+1} = f(x_n)$ yields $L = f(L)$.

We call a point $x \in I$ a **fixed point** of f if and only if $x = f(x)$.

Assignment:
Prove Part 2 of this theorem (Theorem 7).