Lab 2

In this lab, we will practice with Mathematica the following topics:

- Approximate the value of a series within an allowable error.
- Find the radius of convergence and interval of convergence of a power series.
- Approximate a function by polynomials.

# 1 Getting access

There are two ways to get free access to Mathematica:

A) Install three free components: *Wolfram Engine*, *JupyterLab*, and *WolframLanguageForJupyter*. You can use the unlimited computing power of Mathematica on your own computer, with JupyterLab acting as a user interface. The instruction is here:

https://web.engr.oregonstate.edu/~phamt3/Resource/Wolfram-Mathematica-with-JupyterLab.pdf

B) Use the cloud-based version of Mathematica: https://www.wolframcloud.com
In this option, you are limited to about 8 minutes of computation per month. Files stored on the cloud will be deleted after 60 days.

# 2 Approximate the value of a series

A series $\sum a_n$ is an infinite sum. Unless it is of special type such as a geometric series or telescoping series, the task of finding the *exact value* of the series is generally challenging. A calculator can only add finitely many terms, not infinitely many terms. Nevertheless, if given any allowable error, one can *approximate* the series within that error. This is a satisfactory solution for most practical purposes. As a starting example, let us consider an alternating series $\sum(-1)^n b_n$. If the sequence $\{b_n\}$ is decreasing and convergent to 0 then by the Alternating Series Estimation Theorem (page 463 of the textbook),

$$\left|\sum_{n=1}^{\infty}(-1)^n b_n - \sum_{n=1}^{m}(-1)^n b_n\right| \le b_{m+1}.$$

In other words, the sum of the first $m$ terms (i.e. the $m$'th partial sum) differs from the exact value of the series by an error less than $b_{m+1}$. Therefore, to find an approximate value of the series $\sum(-1)^n b_n$ with an allowable error $\epsilon$, we will find a number $m$ (smallest is preferred) such that $b_{m+1} < \epsilon$. Then the $m$'th partial sum $s_m = \sum_{n=1}^{m}(-1)^n b_n$ is an approximate value we need.

For example, we want approximate the value of $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n+n}$ with an allowable error $\epsilon = 10^{-6}$. In this case, $b_n = \frac{1}{2^n+n}$, which is a decreasing sequence convergent to 0.

(1) Hence, we need to find the smallest number $m$ such that $b_{m+1} < 10^{-6}$. You can try plugging $m = 1, 2, 3, 4, \ldots$ into $b_{m+1}$ to see which of them is the first one to make $b_{m+1} < 10^{-6}$. Alternatively, you can use the **While** loop of Mathematica. The syntax of the **While** loop is

While[*test, body*]

Mathematica will check the *test* first. If true, it executes the *body*. Then it checks the *test* again. If true, it will execute the *body* again. And then go back to check the *test*. The loop continues until the *test* is false. *Note that if the body contains multiple commands, these commands have to be separated from each other by the semicolons.* Try the following:

```
b[n_] := 1/(2^n + n);
m = 1;
esp = 10^(-6);
While[b[m+1] > esp, m = m+1]
```

(2) After executing the above, type m and press Shift+Enter to see what the value of $m$ is. This is the smallest value of $m$ such that $b_{m+1} < \epsilon$. The approximate value of the series is $s_m = \sum_{n=1}^{m}(-1)^n b_n$, which is computed by the command

```
S = Sum[(-1)^(n+1)*b[n], {n,1,m}]
```

(3) To see what the fraction is in decimal form, try

```
N[S]
N[S, 8]
N[S, 10]
```

Only 6 digits (after the decimal point) of $S$ are in the exact value of the series because we only computed the series with an error of $10^{-6}$. Any digit after the 6th decimal place is not reliable.

(4) If you want to approximate the above series up to 9 decimal places, how many terms of the series do you need to use? What is the approximation of the series?

The following representation of the number $\pi$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - ... = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

is famously known as the Leibniz's formula. If you are interested in learning more about the history of this formula (quite fascinating), take a look at this article:

https://www-jstor-org.eou.idm.oclc.org/stable/2690896

(5) To use Leibniz's formula to estimate $\pi$ correctly up to 5 decimal places, how many terms of the series should you take? And what is the estimate value of $\pi$ accordingly?

(6) There is another representation of $\pi$

$$\frac{\pi}{2\sqrt{3}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(2n+1)}$$

This formula was found by Abraham Sharp in about 1717. If you are to use this formula to approximate $\pi$ correctly up to 5 decimal places, how many terms in the sum should you use?

(7) Since both Sharp's formula and Leibniz's formula can be used to approximate $\pi$, what are some advantages and disadvantages of one method compared to the other?
*Remark: there are many other series representations of $\pi$. You can find a long, yet incomplete, list of them here:* *https://mathworld.wolfram.com/PiFormulas.html*

## 3 Find radius and interval of convergence

Recall that a *power series centered at* $x_0$ is a series of the form $\sum a_n(x - x_0)^n$. On its interval of convergence, a power series defines a function. Not counting the endpoints, the interval of convergence is always a symmetric interval about the center of the power series. Consider the power series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

To find the radius of convergence, one can use either the Ratio test or the Root test. Let

$$a_n = \frac{(-3)^n x^n}{\sqrt{n+1}}$$

The Ratio test says that if the limit

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

exists and $L < 1$ then the series converges. If $L > 1$ then the series diverges. If $L = 1$ then the test fails.

(8) We compute the limit $L$ as follows.

```
a[n_] := (-3)^n*x^n/Sqrt[n + 1]
Clear[n]
L = Limit[Abs[a[n + 1]/a[n]], n -> Infinity]
```

Here, `Abs` is the absolute value function.

(9) The values of $x$ that makes $L < 1$ belongs to the interval of convergence. We solve the inequality $L < 1$ as follows.

```
Reduce[L < 1, x, Reals]
```

The option `Reals` in the above command is to indicate that we are interested in $x$ as a real number (instead of a complex number).

(10) You will see that the inequality $L < 1$ gives $x \in \left(-\frac{1}{3}, \frac{1}{3}\right)$. The radius of convergence is a half of the length of this interval, which is $R = \frac{1}{3}$. The endpoints $-1/3$ and $1/3$ have to be considered manually and separately. Mathematica can provide some insights as follows. Let $f(x)$ denote the value of the power series.

```
f[x_] := Sum[(-3)^n*x^n/Sqrt[n + 1], {n, 0, Infinity}]
```

We can attempt to evaluate $f$ at $x = 1/3$ and $x = -1/3$.

```
f[1/3]
f[-1/3]
```

Mathematica will show a warning on the second command, indicating that $f$ is not defined at $x = -1/3$. Therefore, the interval of convergence is $(-1/3, 1/3]$.

Next, consider the power series

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$$

(11) What is the center of this power series? Find the radius of convergence.

(12) Find the interval of convergence.

(13) Approximate the value of the power series at $x = 0$.

## 4 Approximate a function by polynomials

In many applications, it is helpful to approximate a function with polynomials. Polynomials are easier to take derivative or integral, and more computer-friendly because they involve only the addition, subtraction, and multiplication. To approximate a function by a polynomial, we simply truncate the Taylor series of the function. Recall that the Taylor series of a function $f$ at $x_0$ is given by

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n, \quad \text{where} \quad a_n = \frac{f^{(n)}(x_0)}{n!}$$

Truncating this series at a power $m$, we get an $m$'th degree Taylor polynomial

$$T_m(x) = \sum_{n=0}^{m} a_n(x-x_0)^n$$

and $f$ is approximated by $f(x) \approx T_m(x)$. The approximation is good when $x$ is close to $x_0$, which is the center of the power series, and gets worse as $x$ is far away from the center. To maintain a good approximation when $x$ is far away from $x_0$, you will have to increase the degree $m$. The larger $m$ is, the farther away $x$ can be from $x_0$ and the approximation is still good.

To obtain the Taylor polynomials $T_m$ centered at $x_0 = 0$ of a function $f$, we use the command **Series** with the syntax

$$\text{Series[f, \{x, } x_0 \text{, m\}]}$$

(14) For example, consider the function $f(x) = \sin x + \cos(x/\sqrt{2})$. Try the command

```
f[x_] := Sin[x] + Cos[x/Sqrt[2]]
Series[f[x], {x, 0, 7}]
```

The output will look something like

$$1 + x - \frac{x^2}{4} - \frac{x^3}{6} + \frac{x^4}{96} + \frac{x^5}{120} - \frac{x^6}{5760} - \frac{x^7}{5040} + O\left(x^8\right)$$

Therefore, the first seven Taylor polynomials $T_1, T_2, ..., T_7$ are

$$\begin{aligned}
T_1(x) &= 1 + x \\
T_2(x) &= 1 + x - \frac{x^2}{4} \\
T_3(x) &= 1 + x - \frac{x^2}{4} - \frac{x^3}{6} \\
&\dots \\
T_7(x) &= 1 + x - \frac{x^2}{4} - \frac{x^3}{6} + \frac{x^4}{96} + \frac{x^5}{120} - \frac{x^6}{5760} - \frac{x^7}{5040}
\end{aligned}$$

The term $O(x^8)$ denotes the error term, which is the difference between the function $f$ and the polynomial $T_7$.

(15) To see how well each Taylor polynomial approximates the function $f$, we graph them together on the same plot. For example, try the following to graph $f$ and $T_1$ on the same plot.

```
T1[x_] := 1+x
Plot[{f[x], T1[x]}, {x,-3,3}]
```

(16) Use the fashion above to graph each of the functions $T_2, T_3, ..., T_7$ (one by one, not all at once) together with $f$ on the same plot. What do you observe?

(17) One way to quantify how good the approximation $f(x) \approx T_m(x)$ is on the interval $x \in [-3, 3]$ is by looking at the maximum value of $|f(x) - T_m(x)|$ on the interval $[-3, 3]$. Try

```
MaxValue[{Abs[f[x] - T1[x]], -3 <= x <= 3}, x]
NMaxValue[{Abs[f[x] - T1[x]], -3 <= x <= 3}, x]
```

(18) For $m = 2, 3, ..., 7$, find the maximum of $|f(x) - T_m(x)|$ on the interval $[-3, 3]$.

(19) How large does $m$ have to be so that $f(x) \approx T_m(x)$ with an error less than 0.01 for any $x \in [-3, 3]$ ?

In Calculus II, you learned how to approximate a definite integral of a function using Riemann sums. That was helpful when you don't know the antiderivative of the function. Here, we learn another method (simpler) to approximate a definite integral using Taylor polynomials. The idea is very simple:

$$\int_a^b f(x)dx \approx \int_a^b T_m(x)dx$$

The integral of $T_m$ is always computable because $T_m$ is a polynomial. To estimate the error of this approximation, we note that

$$\left| \int_a^b f(x)dx - \int_a^b T_m(x)dx \right| \leq \int_a^b |f(x) - T_m(x)|dx \leq (b-a) \max_{x\in[a,b]} |f(x) - T_m(x)|.$$

Therefore, to estimate $\int_a^b f(x)dx$ with an allowable error $\epsilon$, we will

- first, find some Taylor polynomials $T_1, T_2, T_3, ...$ of the function $f$. To get the best approximation, the center $x_0$ of these polynomials should be taken as $\frac{a+b}{2}$, the midpoint of $[a, b]$.
- second, choose $m$ such that

$$\max_{x\in[a,b]} |f(x) - T_m(x)|dx < \frac{\epsilon}{b-a}$$

- finally, compute $\int_a^b T_m(x)dx$. That is an approximate value of $\int_a^b f(x)dx$ that we need.

(20) For example, let us evaluate $\int_0^2 \sin(-x + \cos(x-1))dx$ with a permissible error $\epsilon = 0.001$. Use the following command to see if Mathematica is able to find the exact value of this integral:

```
Integrate[Sin[-x + Cos[x - 1]], {x, 0, 2}]
```

(21) Because the midpoint of the interval $[0, 2]$ is 1, we will approximate the function $f(x) = \sin(-x + \cos(x-1))$ by Taylor polynomials centered at 1. Can you write a command to get the 12th degree polynomial of this function? Define it as a function $T(x)$ using `T[x_]:=...` (copy/paste the output from the previous command)

(22) Now use the command `NMaxValue` to find the maximum of $|f(x) - T(x)|$ on the interval $[0, 2]$.

```
NMaxValue[{Abs[f[x] - T[x]], 0 <= x <= 2}, x]
```

Is it less than $\frac{\epsilon}{b-a} = \frac{0.001}{2-0} = 0.0005$ ? If yes, we can proceed to the next step. If no, try to define $T$ as the 13th, 14th,... degree Taylor polynomials until the maximum is under 0.0005.

(23) Finally, we compute the integral of $T(x)$ .

```
Integral[T[x], {x, 0, 2}]
```

That is the approximate value of the original integral that we were looking for.

(24) Use the procedure outlined above to find an approximate value of the integral

$$\int_{-3}^{1} \cos(e^{2x+2} - 1)dx$$

with permissible error 0.001.

## 5 To turn in

Submit your implementation of Exercises (1) - (24) as a single pdf file.