Lecture 11

Friday, April 21, 2023 2:40 PM

* Questions..

Integral test

Ex. $\sum \frac{1}{n^2-n}$

$$\frac{1}{n^2-n} \le \frac{2}{n^2} \Leftrightarrow n^2 \le 2(n^2-n) \Leftrightarrow 2n \le n^2 \Leftrightarrow 2 \le n$$

which is true if $n \geq 2$.

$$\sum \frac{2}{n^2} = 2\sum \frac{1}{n^2}$$

converges by the p-series test. Thus, $\sum \frac{1}{n^2-n}$ converges by the Comparison Test

One can also solve this problem by directly using the Integral Test

$$a_n = \frac{1}{n^2-n} = f(n), \text{ with } f(x) = \frac{1}{x^2-x}$$

$$f'(x) = \frac{-(2x-1)}{(x^2-x)^2} < 0 \text{ for all } x \geq 2$$

Thus, f is decreasing on $[2,\infty)$. Moreover, $f(x) \geq 0$ on $[2,\infty)$.

$$\int_2^\infty f(x)\,dx = \int_2^\infty \frac{1}{x^2-x}\,dx = \int_2^\infty \left(\frac{1}{x-1} - \frac{1}{x}\right)dx = \left(\ln(x-1) - \ln x\right)\Big|_2^\infty$$

$$= \ln \frac{x-1}{x}\Big|_2^\infty = \ln 2 < \infty$$

Ex. $\sum \frac{\sin n}{n^2}$

We can't apply the Integral test because $f(x) = \frac{\sin x}{x^2}$ is not positive.

Instead, we will use Comparison Test first

$$\left|\frac{\sin n}{n^2}\right| = \frac{|\sin n|}{n^2} \le \frac{1}{n^2}$$

$\sum \frac{1}{n^2}$ converges because it is a 2-series test.

Therefore, $\sum \frac{\sin n}{n^2}$ converges by the Comparison Test.

Alternating series Test

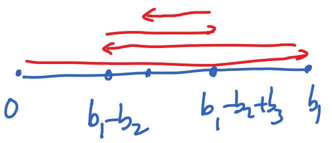
An alternating series is a series of the form

$$b_1 - b_2 + b_3 - b_4 + b_5 - \quad = \sum (-1)^{n+1} b_n$$

or $$-b_1 + b_2 - b_3 + b_4 - b_5 + \cdots = \sum (-1)^n b_n$$

Alternating series test.

If $\{b_n\}$ is decreasing and $\lim b_n = 0$ then $\begin{cases} \sum (-1)^n b_n \\ \sum (-1)^{n+1} b_n \end{cases}$ converges.



Ex $$\sum (-1)^n \frac{1}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \cdots$$

Converges because $b_n = \frac{1}{n}$ is a decreasing sequence and $\lim b_n = 0$.

Ex $$\sum (-1)^n \frac{\sin n}{n}$$

$b_n = \frac{\sin n}{n} \longrightarrow 0$ as $n \to \infty$, but $\{b_n\}$ is not decreasing.

So, we can't use the A.S.T to conclude. Another test is needed.

$\underline{\underline{Ex}}$ $\sum (-1)^n \frac{n}{(n-\pi)^3}$

$b_n = \frac{n}{(n-\pi)^3} = f(n)$, where $f(x) = \frac{x}{(x-\pi)^3}$.

Let's only consider $x > \pi$.

$$f(x) = x(x-\pi)^{-3}$$

$$f'(x) = (x-\pi)^{-3} + x(-3)(x-\pi)^{-4} = (x-\pi)^{-4}(x-\pi-3x)$$

$$= (x-\pi)^{-4}(-2x-\pi) < 0$$

Thus, f is decreasing on $(3, \infty)$.

$$\lim_{x\to\infty} f(x) = 0.$$

Hence, $\{b_n\}$ is decreasing for $n \geq 4$ and $\lim b_n = 0$.

By the A.S.T, the series $\sum (-1)^n b_n$ converges.