Lecture 12

Monday, April 24, 2023 10:21 AM

* Questions.

Approximate a series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \dots \approx a_1 + a_2 + \quad + a_m = S_m$$

The error of this approximation is

$$\sum_{n=1}^{\infty} a_n - S_m = \underbrace{a_{m+1} + a_{m+2} + \dots}_{\text{the tail of the series}}$$

There is no general rule to estimate this tail. However, it has a nice estimate of the series is an alternating series:

$$a_n = (-1)^n b_n$$

Suppose $\{b_n\}$ is a decreasing series and $\lim b_n = 0$.

The tail is

$$\sum_{n=1}^{\infty} (-1)^n b_n - S_m = \underbrace{(-1)^{m+1} b_{m+1} + (-1)^{m+2} b_{m+2} + \dots}_{(*)}$$

Suppose m is odd. Then

$$(*) = \underbrace{b_{m+1} - b_{m+2}}_{\geq 0} + \underbrace{b_{m+3} - b_{m+4}}_{\geq 0} + \dots \geq 0$$

$$(*) = b_{m+1} + \underbrace{(-b_{m+2} + b_{m+3})}_{\leq 0} + \underbrace{(-b_{m+4} + b_{m+5})}_{\leq 0} + \dots \leq 0$$

Therefore, $0 \le (*) \le b_{m+1}$.

If m is even, we get $-b_{m+1} \le (*) \le 0$. We can combine both cases as follows.

$$|(*)| \le b_{m+1} \quad \text{for any } m.$$

In practice, to make sure that the error $|(*)|$ is under certain allowable error ε, we choose m large enough so that $b_{m+1} \le \varepsilon$.

Ex Approximate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n2^n}$ with allowable error $\varepsilon = 10^{-4}$.

Here $b_n = \dfrac{1}{n2^n}$.

$\{b_n\}$ is a decreasing sequence and $\lim b_n = 0$.

The error term

$$\left| \sum_{n=1}^{\infty} (-1)^n b_n - \sum_{n=1}^{m} (-1)^n b_n \right| \le b_{m+1} = \frac{1}{(m+1)2^{m+1}}$$

We want to choose m such that

$$\frac{1}{(m+1)2^{m+1}} < 10^{-4}$$

$m = 9$ will do it. Therefore, within the allowable error 10^{-4},

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n2^n} \approx \sum_{n=1}^{9} \frac{(-1)^n}{n2^n} = \frac{-1}{2} + \frac{1}{2\cdot 2^2} - \frac{1}{3\cdot 2^3} + \frac{1}{4\cdot 2^4} - \frac{1}{5\cdot 2^5}$$

$$+ \frac{1}{6\cdot 2^6} - \frac{1}{7\cdot 2^7} + \frac{1}{8\cdot 2^8} - \frac{1}{9\cdot 2^9}$$

$$\approx -0.40553.$$

Ex Finding many digits of π was a fascinating problem for a long time. Today, it is not so much due to the available technology.

There are many series that can be used to approximate π. Here are some

* Leibnitz's formula:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

* Sharp's formula

$$\frac{\pi\sqrt{3}}{6} = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(2n+1)}$$

* Ramanujan's formula:

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{(2n)!^3}{(n!)^6} \frac{42n+5}{2^{12n+4}}$$

Which of these converges fastest to π? That is, with the same number of term m, which formula gives the best approximation for π? Sharp's formula is better than Leibnitz's formula