4/28/23 **Power Series**

power series: $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$

For what values of x does the series converge?

If the series converges, we get a function
$f(x) = \sum_{n=1}^{\infty} a_n x^n$

Definition:
A power series centered at x_0 is a series of the form $\sum_{n=0}^{\infty} a_n (x - x_0)^n$

Theorem:
There is a $R \in [0, \infty]$ such that $\sum a_n (x - x_0)^n$ converges if $x \in (x_0 - R, x_0 + R)$ and $\sum a_n (x - x_0)^n$ diverges if $x \in (-\infty, x_0 - R) \cup (x_0 + R, \infty)$.

divergence ? convergence ? divergence

$\underbrace{x_0 - R \quad\quad x_0}_{R} \underbrace{\quad\quad x_0 + R}_{R}$

R is the radius of convergence

R is determined using the ratio test or the root test.

* The set of x's where the series converges is called the interval of convergence
* to find center only look at the power

ex. Find the interval of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n}$.

$\sum a_n (x - 0)^n$ centered at 0

Step 1: Determine R

$a_n = \frac{x^n}{n}$

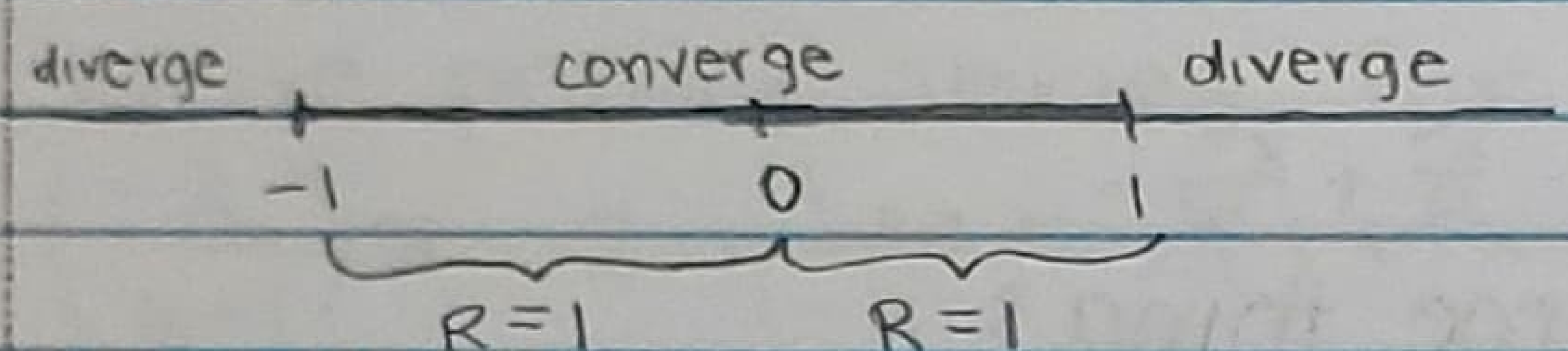
$$\frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}} = \frac{\cancel{x^n} x}{n+1} \cdot \frac{n}{\cancel{x^n}} = \frac{xn}{n+1}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{xn}{n+1}\right| = \frac{|xn|}{|n+1|} = \frac{|x|\,|n|}{|n+1|} = \frac{|x|\,n}{n+1}$$

$$\lim_{n\to\infty} \frac{|x|n}{n+1} = |x|$$

By ratio test:

$\sum \frac{x^n}{n}$ converges if $|x|<1$, diverges if $|x|>1$.



Step 2: check endpoints

Check $x=-1$:

$$\sum \frac{x^n}{n} = \sum \frac{(-1)^n}{n} = \sum (-1)^n \overbrace{\frac{1}{n}}^{b_n}$$

Series converges by alternating series test.

b_n is decreasing, $\lim b_n = 0$.

Check $x=1$:

$$\sum \frac{x^n}{n} = \sum \frac{1}{n}$$

diverges because it is a 1-series

Conclusion:

The intergral of convergence is $[-1,1)$.

ex. $\displaystyle\sum_{n=1}^{\infty} \frac{(2x+1)^{2n}}{9^n}$

$$\frac{(2x+1)^{2n}}{9^n} = \frac{\left[2(x+\frac{1}{2})\right]^{2n}}{9^n} = \frac{2^{2n}(x+\frac{1}{2})^{2n}}{9^n}$$

$$= \frac{2^{2n}}{9^n}\left(x+\frac{1}{2}\right)^{2n}$$ ← looks more like power series

Find interval of convergence.

$a_n = \frac{(2x+1)^{2n}}{9^n}$ $\quad |a_n| = a_n \quad \sqrt[n]{a_n} = \sqrt[n]{\frac{(2x+1)^{2n}}{9^n}} = \frac{(2x+1)^2}{9}$

If $\frac{(2x+1)^2}{9} < 1$, we get convergence

$\Leftrightarrow (2x+1)^2 < 9 = 3^2$

$\Leftrightarrow |2x+1| < 3 \quad \Leftrightarrow -3 < 2x+1 < 3 \quad \Leftrightarrow -2 < x < 1$