

Power Series

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Starting with a familiar function, can we write it as a power series?

$$e^x = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

Plug $x=0$: $e^0 = a_0 + 0 \rightarrow a_0 = 1$

Take derivative: $e^x = 0 + a_1 + 2a_2 x + 3a_3 x^2 + \dots$

Plug $x=0$: $e^0 = a_1 + 0 \rightarrow a_1 = 1$

Take derivative: $e^x = 0 + 0 + 2a_2 + 6a_3 x + \dots$

Plug $x=0$: $e^0 = 2a_2 + 0 \rightarrow 1 = 2a_2 \rightarrow a_2 = \frac{1}{2}$

Take derivative: $e^x = 0 + 0 + 0 + 6a_3 + \dots$

Plug $x=0$: $e^0 = 6a_3 + 0 \rightarrow 1 = 6a_3 \rightarrow a_3 = \frac{1}{6}$

$$\boxed{a_n = \frac{1}{n!}} \quad e^x = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

Thm:

If function $f(x)$ has a power series representation then $f(x) = \sum a_n (x-x_0)^n$ where

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$