5/8/23 Power Series

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$

$e^x \rightarrow$ subsitute x by $x^2 \rightarrow e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$

$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$

$2^x = e^{(\ln 2)x} = \sum_{n=0}^{\infty} \frac{((\ln 2)x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(\ln 2)^n}{n!} x^n$

$\ln(1+x-1)$

$\ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}$

*Recall: If a function f has a power series representation at x_0 then $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ with $a_n = \frac{f^{(n)}(x_0)}{n!}$.

If f has a power series at x_0, that power series is unqiue.

(*) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\quad e^x = \sum_{n=0}^{\infty} a_n (x-1)^n$

How to get a_n? Two Methods:

(1) $a_n = \frac{f^{(n)}(1)}{n!}$

$f(x) = e^x \quad f^{(n)}(x) = e^x \rightsquigarrow a_n = \frac{e^1}{n!} = \frac{e}{n!}$

$e^x = \sum_{n=0}^{\infty} \frac{e}{n!} (x-1)^n$

(2) In (*), replace x by $x-1$

$e^{x-1} = \sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$ Multiply both sides by e

$e^x e^{-1}$ $\quad e^x = e\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!} = \sum_{n=0}^{\infty} \frac{e}{n!} (x-1)^n$

$$f(x) = \sum \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \qquad \text{Taylor (1715)}$$

$\rightsquigarrow$ Taylor series of f at x_0.

When $x_0 = 0$, Taylor series is also known as a Maclaurin series.

Taylor series $\rightarrow$ a power series representation of a function

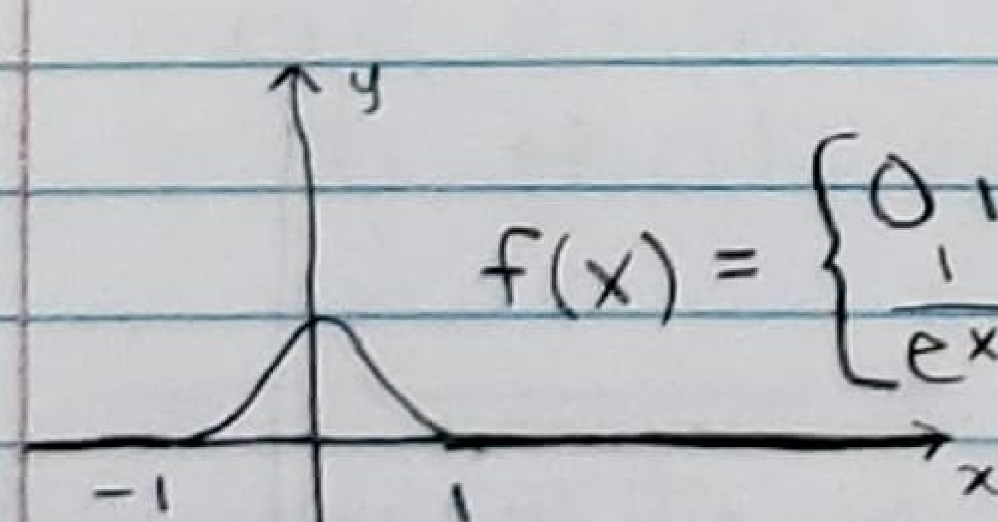
Theorem:

If f admit a Taylor series at x_0, f has to be infinitely differentiable at x_0.

ex. $f(x) = |x|$ doesn't admit a Maclaurin series, because f is not differentiable at 0.

Remark: If a function f is infinitely differentiable at x_0, it may or may not admit a power series representation at x_0.

ex.



$$f(x) = \begin{cases} 0 & \text{if } |x| \geq 1 \\ \frac{1}{e^{x^2-1}} & \text{if } |x| < 1 \end{cases}$$

f is infinitely differentiable at $x = 1$ and $x = -1$

$$f(x) = \sum \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum 0 \cdot (x-1)^n = 0$$