5/9/23 Power Series

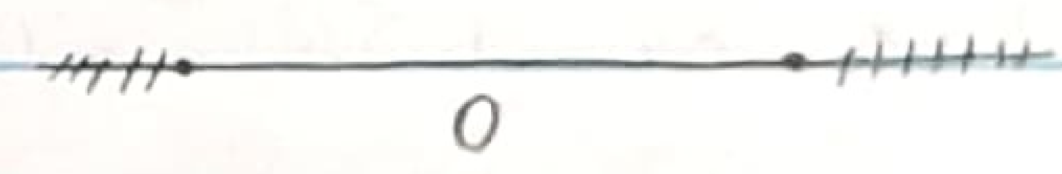
$\sum a_n x^n = f(x)$

$f(x) := \sum a_n x^n$

$f(x) = \ln x, e^x, \sin x, \sin(\ln x), \ldots$

$f(x) = \sum a_n x^n \qquad a_n = \frac{f^{(n)}(x_0)}{n!}$ (*)

$\frac{1}{1-x} = \sum a_n x^n$



Match on $(-1, 1)$

- The formula (*) is useful when the n'th derivative of f is easy to compute.

$f(x) = e^x$

$f(x) = 1 + 5x + x^2 + 9x^3 - 16x^4 \ldots$ (already power series)

$f(x) = \frac{1}{1-x}$

$f(x) = \sin x \qquad f''(x) = -\sin x \qquad f^{(4)}(x) = \sin x$

$f'(x) = \cos x \qquad f'''(x) = -\cos x \qquad f^{(5)}(x) = \cos x$

$f^{(6)}(x) = -\sin x \qquad f^{(7)}(x) = -\cos x$

$a_0 = \frac{f(0)}{0!} = 0 \qquad a_3 = \frac{f^{(3)}(0)}{3!} = \frac{-1}{3!}$

$a_1 = \frac{f'(0)}{1!} = 1 \qquad a_4 = 0$

$a_5 = \frac{1}{5!}$

$a_2 = \frac{f''(0)}{2!} = 0$

odd function $\sin x = \cancel{a_0} + a_1 x + \cancel{a_2 x^2} + a_3 x^3 + \cancel{a_4 x^4} + a_5 x^5 + \ldots$

$= \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots$

Euler used this power series to show that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \frac{\pi^2}{6}$

even function $\cos x = \frac{1}{1!} - \frac{2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$

$(1+x)^6 = 1 + \frac{6}{1}x + \frac{6\cdot5}{1\cdot2}x^2 + \frac{6\cdot5\cdot4}{1\cdot2\cdot3}x^3 + \frac{6\cdot5\cdot4\cdot3}{1\cdot2\cdot3\cdot4}x^4 + \frac{6\cdot5\cdot4\cdot3\cdot2}{1\cdot2\cdot3\cdot4\cdot5}x^5 + \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{1\cdot2\cdot3\cdot4\cdot5\cdot6}x^6$

$= 1 + 6x \cdot 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$

$(1+x)^{1/2} = \sum \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}-k+1)}{k!} x^k$

If the n'th derivative of f is complicated we need to find another way to get the power series.

ex. $f(x) = x\sin x$

$f'(x) = \sin x + x\cos x$

$f''(x) = \cos x + \cos x - x\sin x$

$f^{(3)}(x) = -\sin x - \sin x - \sin x - x\cos x$ more complicated!

Alternative:

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}(-1)^n$

$x\sin x = x\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+2}}{(2n+1)!}$

Taylor's thm:

$\underbrace{T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k}_{\text{n'th degree taylor polynomial}}$

$f(x) = \sum_{n=0}^{\infty} a_n x^n$

$f(x) \approx \underbrace{\sum_{n=0}^{m} a_n x^n}_{\text{polynomial}}$

There exists a number c between x and x_0 such that $f(x) - T_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}$

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* Binomial Series

8.7 * for the hw

$(1+x)^2 = 1 + 2x + x^2 = 1 + \frac{2}{1}x + \frac{2\cdot1}{1\cdot2}x^2$

$(1+x)^3 = 1 + 3x + 3x^2 + x^3 = 1 + \frac{3}{1}x + \frac{3\cdot2}{1\cdot2}x^2 + \frac{3\cdot2\cdot1}{1\cdot2\cdot3}x^3$

$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4 = 1 + \frac{4}{1}x + \frac{4\cdot3}{1\cdot2}x^2 + \frac{4\cdot3\cdot2}{1\cdot2\cdot3}x^3 + \frac{4\cdot3\cdot2\cdot1}{1\cdot2\cdot3\cdot4}x^4$

$(1+x)^n = \sum_{k=0}^{n} \underbrace{\frac{n(n-1)\cdots(n-k+1)}{k!}}_{\text{denoted as } \binom{n}{k}} x^k$ ← Binomial formula