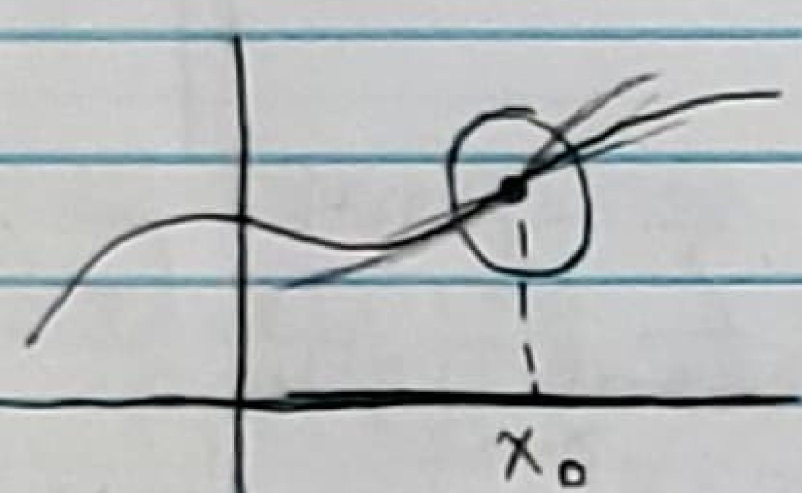
5/11/23 Taylor Theorem

Motivation:
We want to approximate a function by a polynomial.

$f(x) = a_0 + a_1 x + \dots + a_n x^n$ n'th degree polynomial

Which polynomial is the best?
"Best" means: around a given point x_0, the polynomial is closet to f.

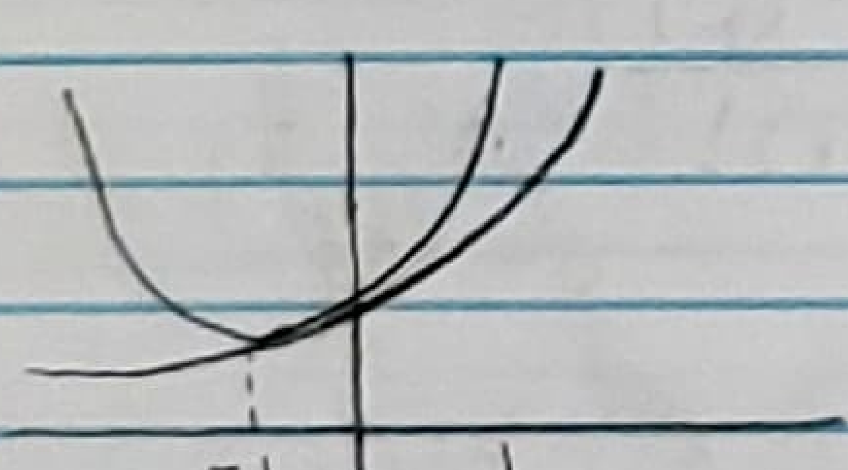


Turns out: best polynomial is
$$T_n(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

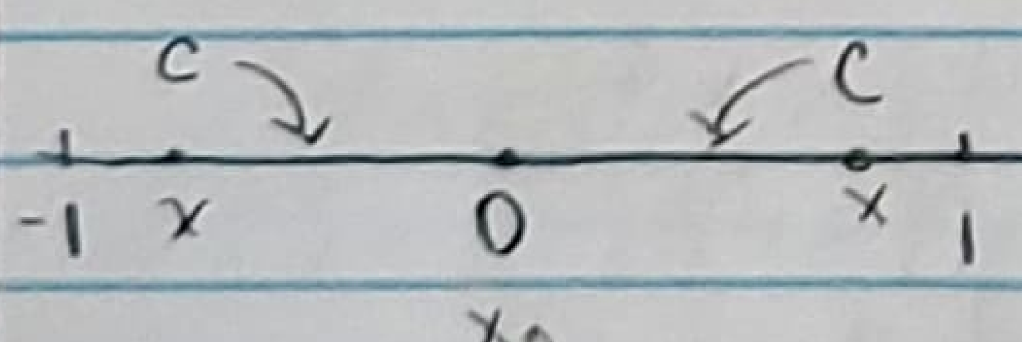
Taylor Thm:
Given function f, number x_0, x, and n, there exist a number c between x and x_0 so that $f(x) - T_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$

axis symmetry
ax^2+bx+c
$x = \frac{-b}{2a}$

ex. Estimate the error of the approximation $e^x \approx 1 + x + \frac{x^2}{2}$ when $x \in (-1,1)$.



$n=2$
$x_0 = 0$
Taylor Thm: there exist c between x and $x_0 = 0$ such that
$$\underbrace{e^x}_{f(x)} - \left(1 + x + \frac{x^2}{2}\right) = \frac{f^{(3)}(c)}{3!}x^3 = \frac{e^c}{6}x^3$$



$$\left|e^x - \left(1 + x + \frac{x^2}{2}\right)\right| = \left|\frac{e^c}{6}x^3\right| = \frac{e^c}{6}|x^3| \le \frac{e^1}{6}\cdot 1^3 = \frac{e}{6} \approx 0.453$$