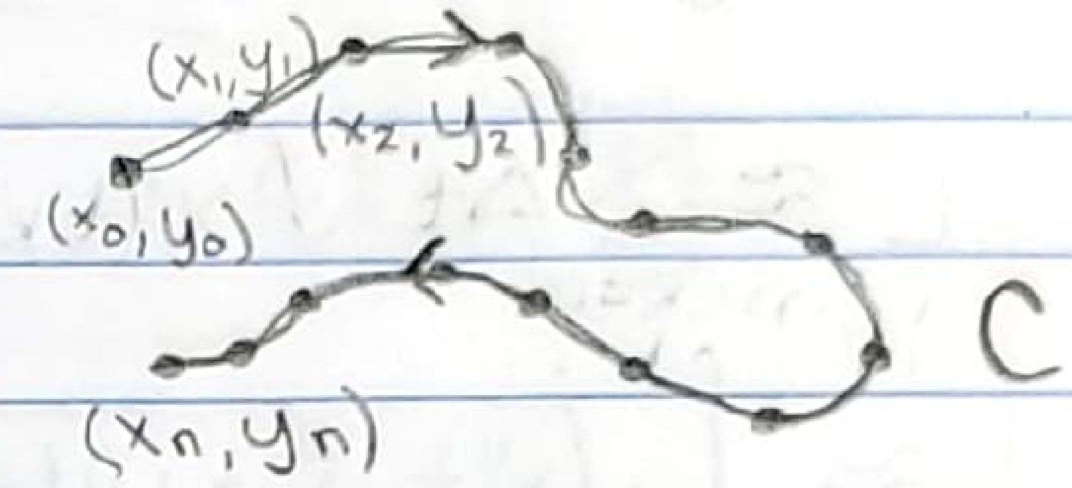
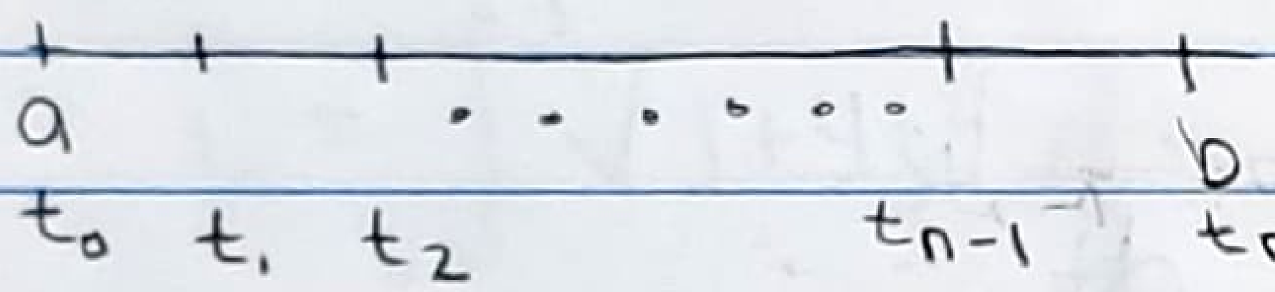
Curves 5/18/23

* Length of a Curve

$$C:\begin{cases} x = x(t) \\ y = y(t) \end{cases} \quad t \in [a, b]$$



Take the gridpoints $t_0 = a < t_1 < t_2 < \ldots < t_n = b$



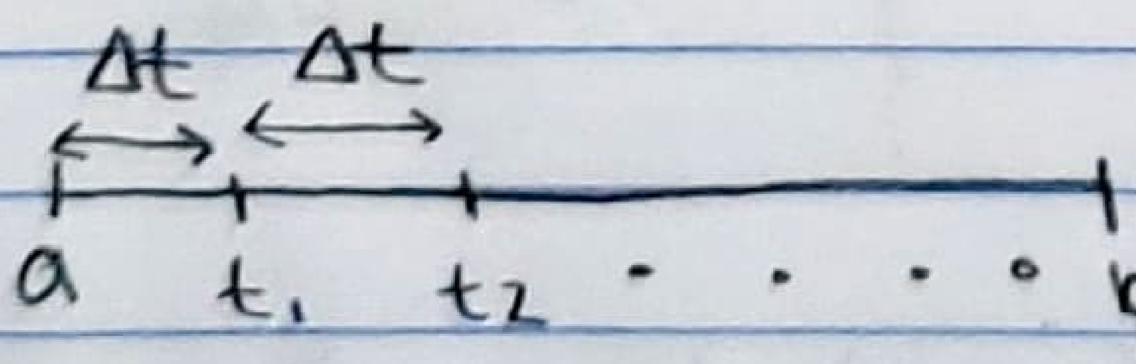
$x(t_n) = x_n$
$y(t_n) = y_n$

The length of the curve is

$L \approx$ distance between (x_0, y_0) and (x_1, y_1)
$+$ " " (x_1, y_1) and (x_2, y_2)
$+$ " " (x_2, y_2) and (x_3, y_2)
$+$ " " (x_{n-1}, y_{n-1}) and (x_n, y_n)

$$= \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2} + \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} + \ldots + \sqrt{(x_{n-1} - x_n)^2 + (y_{n-1} - y_n)^2}$$

$$= \sum_{k=0}^{n-1} \sqrt{(x_{k+1} - x_k)^2 + (y_{k+1} - y_k)^2}$$



$$= \sum_{k=0}^{n-1} \Delta t \sqrt{\frac{(x_{k+1} - x_k)^2 + (y_{k+1} - y_k)^2}{(\Delta t)^2}} \quad (*)$$

$$\frac{(x_{k+1} - x_k)^2 + (y_{k+1} - y_k)^2}{(\Delta t)^2} = \left(\frac{x_{k+1} - x_k}{\Delta t}\right)^2 + \left(\frac{y_{k+1} - y_k}{\Delta t}\right)^2$$

$$= \left(\frac{x(t_{k+1}) - x(t_k)}{t_{k+1} - t_k}\right)^2 + \left(\frac{y(t_{k+1}) - y(t_k)}{t_{k+1} - t_k}\right)^2$$

$$\approx x'(t_k)^2 + y'(t_k)^2$$

$\approx x'(t_k)^2 + y'(t_k)^2$

Plug into (*)

$$L \approx \sum_{k=0}^{n-1} \Delta t \sqrt{x'(t_k)^2 + y'(t_k)^2}$$ Riemann Sum of this function

$$\approx \int_a^b \sqrt{x'(t_k)^2 + y'(t_k)^2}\, dt$$

Formula:

$$L = \int_a^b \sqrt{(x')^2 + (y')^2}\, dt$$

ex. Find the length of the curve

$$C: \begin{cases} x = \frac{4\sqrt{2}}{3} t^{3/2} \\ y = t^2 - t \end{cases}, \quad t \in [0, 2]$$

$x' = 2\sqrt{2}\, t^{1/2} \quad y' = 2t - 1$

$$L = \int_0^2 \sqrt{(2\sqrt{2}t^{1/2})^2 + (2t-1)^2}\, dt$$

$$= \int_0^2 \sqrt{8t + 4t^2 - 4t + 1}\, dt$$

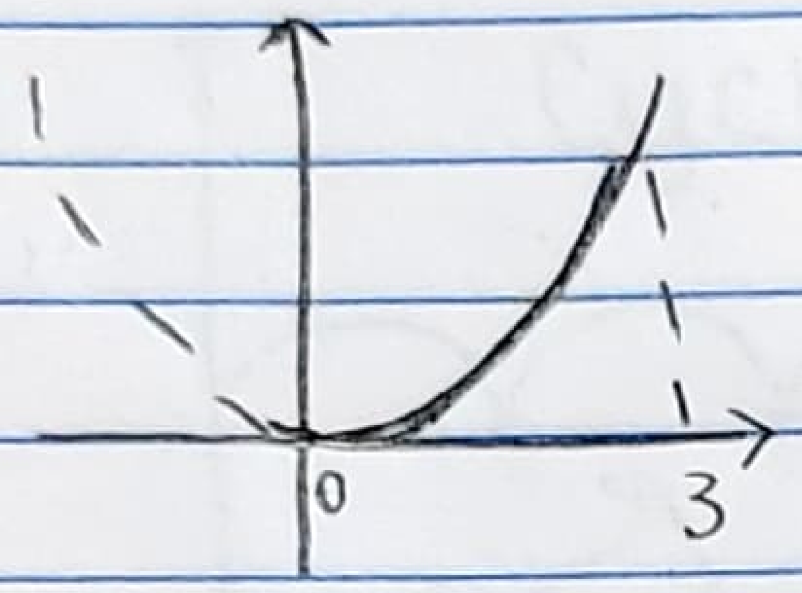
$$= \int_0^2 \sqrt{4t^2 - 4t + 1}\, dt$$

$$= \int_0^2 \sqrt{(2t+1)^2}\, dt = \int_0^2 (2t+1)\, dt$$

$$= t^2 + t \Big|_0^2 \quad (4+2) - (0+0) = \boxed{6}$$

ex. Find the length of the parabola

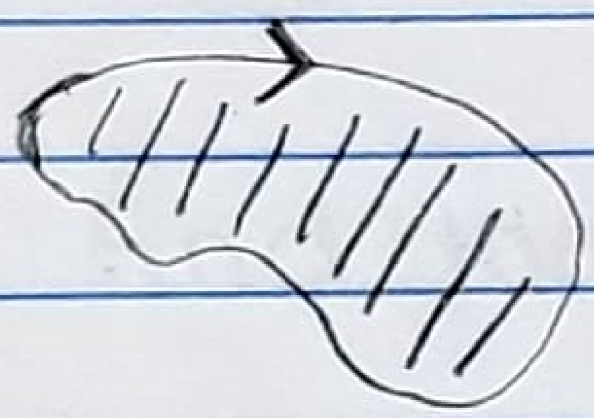
$y = x^2 \quad x \in [0, 3]$



$$\begin{cases} x = t \\ y = t^2 \end{cases} \quad t \in [0, 3]$$

$$(x')^2 + (y')^2 = 1 + (2t)^2 = 1 + 4t^2$$

$$L = \int_0^3 \sqrt{1+4t^2}\, dt$$ (use the table of integral)



closed curve

simple curve

$$C: \begin{cases} x = x(t) \\ y = y(t) \end{cases} \quad t \in [a, b]$$

Area enclosed by the curve is

$$A = \left| \int_a^b x y' \, dt \right|$$