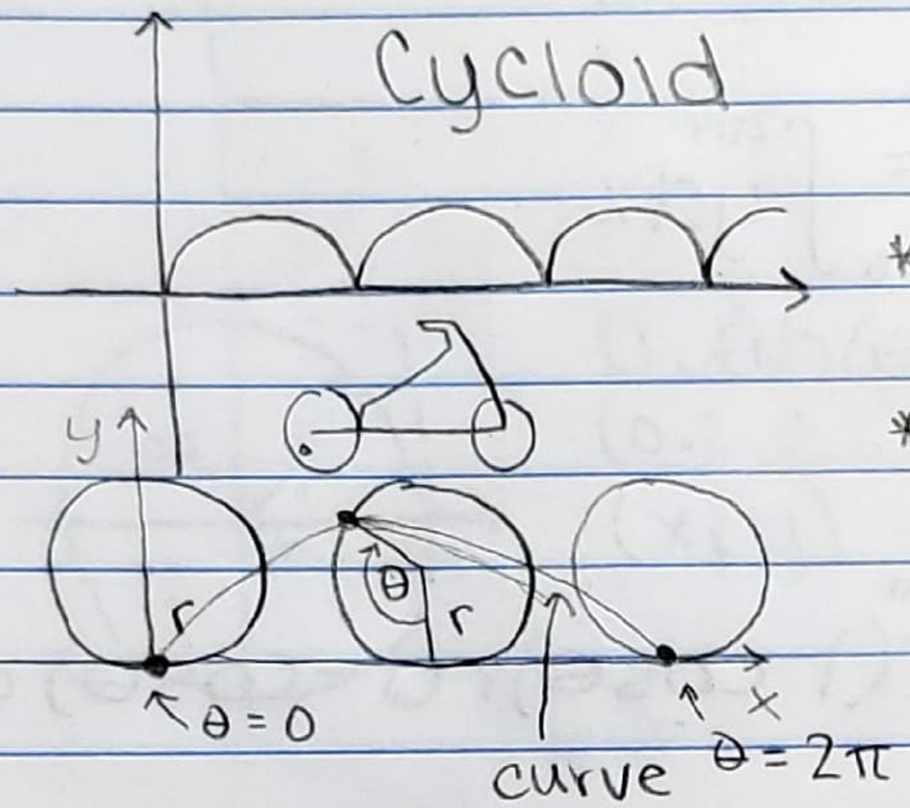
5/19/23

Find Area



Cycloid $\begin{cases} x = r(\theta - \sin\theta) \\ y = r(1 - \cos\theta) \end{cases}$

*where r is the radius of the wheel

* θ is the angle that the wheel has rotated

(a) Find the length of the curve. $\theta \in [0, 2\pi]$

$$(r(\theta - \sin\theta)')^2 + (r(1-\cos\theta)')^2$$
$$(r(1-\cos\theta))^2 + (r\sin\theta)^2$$
$$r^2(1-\cos\theta)^2 + r^2\sin^2\theta$$
$$r^2(1-2\cos\theta) + \underbrace{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$r^2(\cos^2\theta + \sin^2\theta) = r^2$$

$$r^2(1-2\cos\theta) + r^2 = r^2(2-2\cos\theta)$$
$$= 2r^2(1-\cos\theta)$$

Double Angle Identity

$$1-\cos\theta = 2\sin^2\left(\frac{\theta}{2}\right)$$

$$= 2r^2\left(2\sin^2\left(\frac{\theta}{2}\right)\right)$$

$u = \frac{\theta}{2} = \frac{1}{2}\theta$

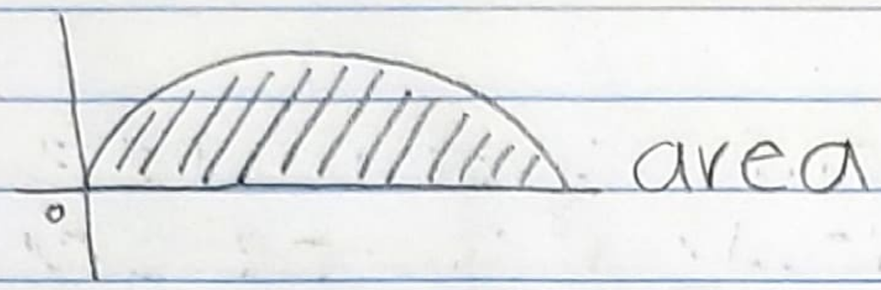
$du = \frac{1}{2}d\theta$

$d\theta = 2du$

$$\int_0^{2\pi} \sqrt{2r^2\left(2\sin^2\left(\frac{\theta}{2}\right)\right)}\, d\theta = \int_0^{2\pi} 2r\left(\sin\left(\frac{\theta}{2}\right)\right) d\theta$$

$$2r\int_0^{2\pi} \sin u\,(2du) = 4r\int_0^{\pi} \sin u = 4r(-\cos u)\Big|_0^{\pi}$$

$$4r(-\cos(\pi)) - 4r(-\cos(0)) = 4r + 4r = 8r$$



(b) find the area

$y = y(x)$ $\quad A = \int_0^{2\pi r} y\,dx$

$dx = x'd\theta = r(1-\cos\theta)d\theta$

x	0	$2\pi r$
θ	0	2π

$$A = \int_0^{2\pi} yx'd\theta = \int_0^{2\pi} r(1-\cos\theta)r(1-\cos\theta)\,d\theta$$

$$r^2\int_0^{2\pi}(1-\cos\theta)^2\,d\theta$$

$$(1-\cos\theta)^2 = 1-2\cos\theta+\cos^2\theta$$

$$1-2\cos\theta+\frac{1+\cos(2\theta)}{2}$$