Lecture 3

Thursday, April 6, 2023 1:18 AM

* Questions...

Limit of a sequence: $\lim a_n = L$

<u>ε-N definition:</u> for each $\varepsilon > 0$, there is N such that

$$|a_n - L| < \varepsilon \quad \text{for all } n > N.$$

Show the Geogebra applet at https://www.geogebra.org/m/fmpsrfhh

Finding the limit of a sequence can benefit from finding the limit of a sequence.

<u>Ex</u>

$$\lim_{n\to\infty} \frac{n}{e^n} = \lim_{x\to\infty} \frac{x}{e^x} \overset{L'H}{=} \lim_{x\to\infty} \frac{1}{e^x} = 0$$

<u>Ex</u>

$$\lim_{n\to\infty} \frac{n - \ln n}{\sqrt{n}} = \lim_{n\to\infty} \left(\sqrt{n} - \frac{\ln n}{\sqrt{n}}\right)$$

$$\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = \lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} \overset{L'H}{=} \lim_{x\to\infty} \frac{1/x}{1/2\sqrt{x}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0,$$

Thus, $\lim_{n\to\infty} \left(\sqrt{n} - \frac{\ln n}{\sqrt{n}}\right) = \infty - 0 = \infty$

<u>Squeeze theorem</u> If $b_n \le a_n \le c_n$ starting from some index. and $\lim b_n = \lim c_n = L$

then $\lim a_n = L$.

$$\begin{tikzcd} b_n \arrow[drr] & \le & a_n \arrow[d, dashed] & \le & c_n \arrow[dll] \\ & & L & & \end{tikzcd}$$

Ex $\quad a_n = \dfrac{\cos n}{n}$

$$\begin{tikzcd} -\frac{1}{n} \arrow[drr] & \le & a_n & \le & \frac{1}{n} \arrow[dll] \\ & & 0 & & \end{tikzcd}$$

Thus, $a_n \to 0$ as $n \to \infty$.

Ex $\quad a_n = \dfrac{(-2)^n}{n!}$

$$-\frac{2^n}{n!} \le a_n \le \frac{2^n}{n!}$$

If we could show that $\lim \dfrac{2^n}{n!} = 0$ then $\lim a_n = 0$.

$$\frac{2^n}{n!} = \frac{2 \cdot 2 \cdot 2 \cdots \cdot 2}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n} \le 2\,\frac{2}{n} = \frac{4}{n} \longrightarrow 0$$

Therefore, $\lim \dfrac{2^n}{n!} = 0$.

Sometimes, it is easier to show that the limit exists rather than compute it.

- If the sequence is increasing and bounded from above then it has a limit.
- If the sequence is decreasing and bounded from below then it has a limit.

Ex Show that the sequence $a_n = n e^{-n}$ is a decreasing sequence.

$$f(x) = x e^{-x}.$$

$$f'(x) = e^{-x} - x e^{-x} = (1-x) e^{-x} \leq 0$$

f is a decreasing function. Thus, a_n is a decreasing function.