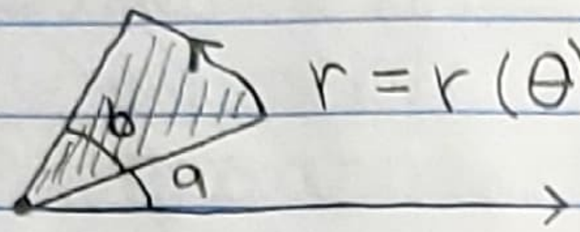
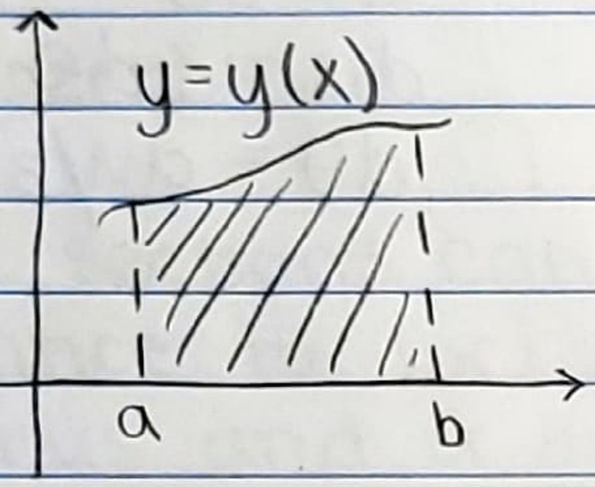
Finding Area Enclosed by a Polar Curve

5/26/23

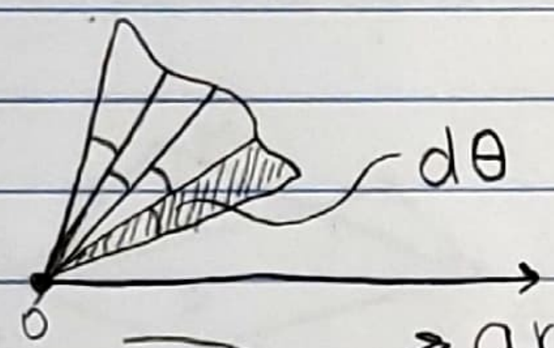


$r = r(\theta)$ $\quad a \le \theta \le b$

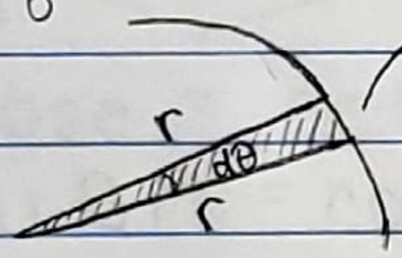


$$\text{area} = \int_a^b y\,dx$$

* Calculus 2



Idea: Break the area into many circular sectors.



$\text{area} = \frac{1}{2}r^2 d\theta$

If $d\theta = 2\pi$, $\text{area} = \pi r^2$

$d\theta = \pi$, $\text{area} = \frac{\pi r^2}{2}$

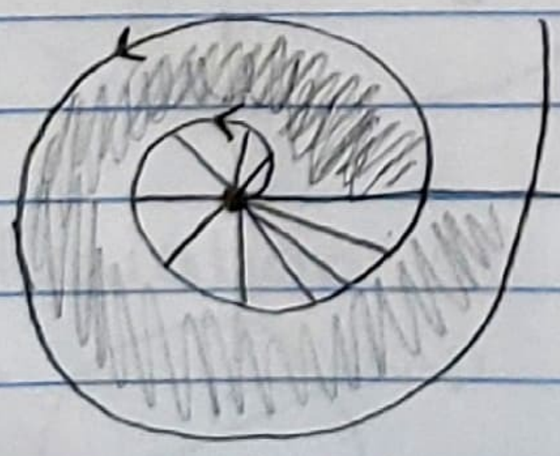
$d\theta = \pi/4$, $\text{area} = \frac{\pi r^2}{8}$

$d\theta = a$, $\text{area} = x$

$$\frac{a}{2\pi} = \frac{x}{\pi r^2} \rightsquigarrow x = \frac{1}{2}r^2 a$$

$$\text{Area} = \int_\alpha^b \frac{1}{2}r^2\,d\theta$$

ex.

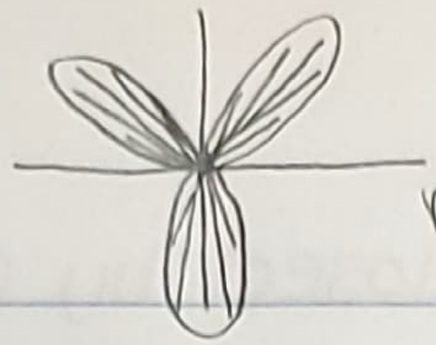


Spiral $r = \theta$ $\quad 0 \le \theta \le 2\pi$

Find area swept by r.

$$\frac{1}{2}\int_0^{2\pi} \theta^2 d\theta = \frac{1}{2}\left(\frac{\theta^3}{3}\right)\Big|_0^{2\pi} \quad \frac{1}{2}\left(\frac{8\pi^3}{3}\right) = \boxed{\frac{4\pi^3}{3}}$$

$$\frac{1}{2}\int_{2\pi}^{4\pi} \theta^2 d\theta - \frac{1}{2}\int_0^{2\pi} \theta^2 d\theta = \frac{28\pi^3}{3} - \frac{4\pi^3}{3} = \frac{24\pi^3}{3} = 8\pi^3$$



$r = \sin 3\theta$

$$\frac{1}{2}\int_0^{2\pi} \sin^2(3\theta)\,d\theta =$$

$$\frac{1}{2}\int_0^{6\pi} \sin^2 u\,d\theta$$

$$\frac{1}{6}\int_0^{6\pi} 1-\cos^2 u\,du$$

$$\frac{1}{6}(u$$

$u = \sin(3\theta)$

$du = 3\cos(3\theta)\,d\theta$

$d\theta = \dfrac{du}{3\cos(3\theta)}$

$u = 3\theta$

$du = 3\,d\theta$

$d\theta = du/3$

$r = \underbrace{\sin 3\theta}$,
periodic with period $\frac{2\pi}{3}$

$0 \le \theta \le \frac{2\pi}{3}$