Lab 2

In this lab, we will practice with Mathematica the following topics:

- Solve separable equations using the *contour method*.
- Predict the future population using Verhurst's model.
- Solve some second order ODEs with `DSolve`.

1 Getting access

There are two ways to get free access to Mathematica:

A) Install three free components: *Wolfram Engine*, *JupyterLab*, and *WolframLanguageForJupyter*. You can use the unlimited computing power of Mathematica on your own computer, with JupyterLab acting as a user interface. The instruction is here:

https://web.engr.oregonstate.edu/∼phamt3/Resource/Wolfram-Mathematica-with-JupyterLab.pdf

B) Use the cloud-based version of Mathematica: https://www.wolframcloud.com
In this option, you are limited to about 8 minutes of computation per month. Files stored on the cloud will be deleted after 60 days.

2 Solve separable equations using the contour method

Recall that the command `DSolve` can be used to solve an ODE with or without an initial condition.

(1) For example, to solve the initial value problem

$$y' = \frac{x^2+1}{y^2+1}, \quad y(0) = 1$$

try the following command

```
DSolve[{y'[x] == (x^2 + 1)/(y[x]^2 + 1), y[0] == 1}, y[x], x]
```

(2) Graph the solution found above on the interval $x \in [0, 3]$. If you forget how to use the command `Plot`, review Exercises 15-17 of Lab 1.
For students working on JupyterLab: use the command `CopyToClipboard[%]` *to copy the output of Exercise 1 and paste it into the next input cell.*

In general, a separable equation only gives solutions in an implicit form.

(3) For example, to solve the equation

$$y' = \frac{x^3+1}{y^3+1}$$

as normally done by hand, we separate y and x as

$$(y^3+1)dy = (x^3+1)dx$$

and then integrate each side:

```
Integrate[y^3+1, y]
Integrate[x^3+1, x]
```

You get an implicit formula $\frac{y^4}{4} + y = \frac{x^4}{4} + x + C$, or equivalently

$$\frac{y^4}{4} + y - \frac{x^4}{4} - x = C.$$

It is difficult, sometimes impossible, to derive an explicit formula for y from the implicit formula.

(4) Suppose the initial condition is $y(0) = 1$. Use the command `DSolve` as in Exercise 1 to see if Mathematica is able to find an explicit formula for the solution. If it is taking too long (more than a 30 seconds), press the combination *Alt + .* to terminate the evaluation.

Whether you get an explicit formula or not, there is a way to visualize the solution from the implicit formula. The command `ContourPlot` plots all the points whose coordinates (x, y) satisfy an equation.

(5) Suppose the initial condition is $y(0) = 1$. In this case, the constant $C = \frac{5}{4}$. To draw the collection of all the points (x, y) satisfying the equation $\frac{y^4}{4} + y - \frac{x^4}{4} - x = \frac{5}{4}$, try the following:

```
ContourPlot[y^3/3 + y - x^3/3 - x == 5/4, {x, -3, 3}, {y, -3, 3},
            ContourStyle -> {Thick, Red}]
```

What you see is the graph of y as a function of x (without knowing an explicit formula of y).

(6) Let us compare the solutions with different initial conditions: $y(0) = 1$ and $y(0) = 2$. In the first case, $C = \frac{5}{4}$. In the second case, $C = 6$. You will plot each solution using `ContourPlot` and then "combine" them into one plot.

```
p1 = ContourPlot[y^3/3 + y - x^3/3 - x == 5/4, {x, -3, 3}, {y, -3, 3},
        ContourStyle -> {Thick, Red}]
p2 = ContourPlot[y^3/3 + y - x^3/3 - x == 6, {x, -3, 3}, {y, -3, 3},
        ContourStyle -> {Thick, Blue}]
Show[p1, p2]
```

(7) Derive an implicit formula for the solution to the ODE

$$y' = \frac{y(5x - 2)}{x(1 - 3y)}$$

given that both x and y are positive.

(8) Suppose the initial condition is $y(1) = 1$. Use `ContourPlot` to visualize the solution.

(9) You can see that the picture shown in the previous exercise is not a graph of a function (Vertical Line Test fails). Can you identify from the picture the largest interval of x on which y is a function of x ? This is called the *maximal interval of existence* of the solution.

(10) Show the solution with the initial condition $y(1) = 1$ and the solution with the initial condition $y(1) = 1/2$ on the same plot. How are they compared to each other?

3 Predict the future population using Verhurst's model

Viewing the unit of population as million or billion, it makes sense to say that population is 0.7 or 1.9 or some other decimal-point number. Verhurst's model for population growth (also known as the *logistic model*) is the following ODE:

$$x' = kx\left(1 - \frac{x}{M}\right)$$

where $x = x(t)$ is the population at time t, M is the population capacity (also known as the maximum sustainable size of the population), k is a constant representing the rate of growth (the difference between the birthrate and the deathrate).

(11) Suppose that the initial population is $x(0) = a$. Use `DSolve` to solve the initial value problem:

```
Clear[x]
DSolve[{x'[t] == k*x[t] (1 - x[t]/M), x[0] == a}, x[t], t]
```

Let $b = e^k$. You can see that the solution found above can be written as

$$x(t) = \frac{aMb^t}{ab^t + M - a}$$

In practice, while a (the initial population) is usually known, the parameters b and M are the "hidden" parameters to be determined. We need to know the population at two other times, $x(t_1)$ and $x(t_2)$, to determine b and M.

(12) Suppose that $x(0) = a = 1$, $x(1) = 3$, $x(2) = 7$. You can solve for b and M using the command `Solve`.

```
a = 1;
x[t_] := a*M*b^t/(a*b^t + M - a);
Solve[x[1] == 3 && x[2] == 7, {b, M}]
```

(13) Now you get an explicit solution x as a function of t. Try the following commands to graph the function $x = x(t)$.

```
b = 7/2;
M = 15;
Plot[x[t], {t, 0, 6}]
```

(14) To visualize the population capacity M and the graph of $x = x(t)$ on the same plot, try the following:

```
p1 = Plot[x[t], {t, 0, 6}]
p2 = Plot[M, {t, 0, 6}, PlotStyle -> Red]
Show[p1, p2]
```

(15) Now we can use this model to predict the population at time $t = 7$. Simply execute the command `x[7]`.

We see that all the unknown parameters in Verhurst's model are completely determined if we know the population at three different times (the initial time and two other times). Now let us test Verhurst's model on the United States population using the census data at https://www.census.gov/data/tables/time-series/dec/popchange-data-text.html

(16) Use the population of the years 1910, 1930, 1950 to determine the population function $x(t)$. View 1910 as the initial time $t = 0$. The year 1930 and 1950 correspond to $t = 20$ and $t = 40$, respectively. What is the population capacity M predicted by the model? Is it reasonable? Graph the function $x(t)$. Does the model give a good prediction for the population in 2020? If not, why do you think the model fails?

(17) Use the population of the years 1950, 1970, 1990 to determine the population function $x(t)$. View 1950 as the initial time $t = 0$. Does the model give a better prediction for the population in 2020 compared to the previous problem? Can you think of a non-mathematical explanation for that? What is the projected population in 2030 according this this model?

(18) Pick three different years of your choice, which you think will give the best estimate for the 2030 population. What is the projected population in 2030?

4 Solve second order ODEs with DSolve

The command `DSolve` can be used to solve second order ODEs.

(19) For example, to get the general solution for the ODE $x'' + 3x' + 2x = 0$, try the following

```
Clear[x]
DSolve[x''[t] + 3 x'[t] + 2 x[t] == 0, x[t], t]
```

(20) To solve the initial value problem

$$x'' + 3x' + 2x = \sin(t), \quad x(0) = 1,\ x'(0) = 2,$$

try the following

```
DSolve[{x''[t] + 3 x'[t] + 2 x[t] == Sin[t], x[0] == 1, x'[0] == 2}, x[t], t]
```

(21) Graph the solution found above.

(22) Solve the initial value problem in Problem 12.1 (ii) on page 118. Then graph the solution. How does the solution behave when t is large?

(23) Solve the initial value problem in Problem 12.1 (iii) on page 118. Then graph the solution. How does the solution behave when t is large?

(24) Solve the initial value problem in Problem 12.1 (iv) on page 118. Then graph the solution. How does the solution behave when t is large?

5 To turn in

Submit your implementation of Exercises (1) - (24) as a single pdf file.