Lecture 10

Thursday, April 20, 2023 1:47 AM

* Questions ...

Continue the problem on estimating the time of death:

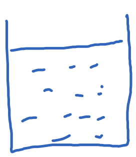
$$\frac{8-a}{9-a} = \frac{\ln(10/30)}{\ln(14/30)} = k$$

$$\rightsquigarrow \quad 8-a = k(9-a) = 9k - ka$$

$$\rightsquigarrow \quad (k-1)a = 9k-8$$

$$\rightsquigarrow \quad a = \frac{9k-8}{k-1} \approx 5.73\ldots \approx 5{:}45 \text{ AM}$$

The environment's temperature is not always a constant.



$T(0) = 200$

$T_0 = \sin t$ (periodic pattern of environment temperature)

$T(1) = 180$

$T(2) = ?$

By Newton's Law of Cooling,

$$T' = \alpha(T - T) = \alpha(\sin t - T)$$

$$\rightsquigarrow \quad T' + \alpha T = \alpha \sin t$$

$$u = e^{\alpha t}$$

$$T = \frac{1}{u}\int uq\,dt = e^{-\alpha t}\underbrace{\int e^{\alpha t}\alpha \sin t\,dt}_{I}$$

Recall integration by parts:

$$\int u'v\,dt = uv - \int uv'\,dt$$

Let $u' = e^{\alpha t}\alpha \rightsquigarrow u = e^{\alpha t}$

$v = \sin t \rightsquigarrow v' = \cos t$

$$I = e^{\alpha t}\sin t - \int e^{\alpha t}\cos t\,dt$$

Let $u' = e^{\alpha t} \rightsquigarrow u = \frac{e^{\alpha t}}{\alpha}$

$v = \cos t \longrightarrow v' = -\sin t$

$$I = e^{\alpha t}\sin t - \left(\frac{e^{\alpha t}}{\alpha}\cos t - \int \frac{e^{\alpha t}}{\alpha}(-\sin t)\,dt\right)$$

$$= e^{\alpha t}\sin t - \frac{e^{\alpha t}}{\alpha}\cos t - \frac{1}{\alpha} I$$

$$\rightsquigarrow I = e^{\alpha t}\frac{\sin t - \frac{\cos t}{\alpha}}{1+\frac{1}{\alpha}} + C$$

Therefore,

$$T = \underbrace{\frac{\sin t - \frac{\cos t}{\alpha}}{1 + \frac{1}{\alpha}}}_{\substack{\text{periodic} \\ \text{with period} \\ 2\pi}} + \underbrace{Ce^{-\alpha t}}_{\substack{\text{decays as } t \to \infty \\ \downarrow \\ 0}}$$

Note:

$$\sin t - \frac{\cos t}{\alpha} = \sin(t - \underbrace{\phi}_{\text{phase shift}})$$

The temperature in the glass also changes periodically with the same period as the temperature of the environment, but is "delayed" by a phase-shift ϕ.

Next time, we will discuss another application of 1st order linear ODE, namely the radioactive decay and carbon dating.