Lecture 12

Monday, April 24, 2023 10:21 AM

* Questions.

We will discuss the existence and uniqueness of solutions. Let's start with the following example

$$\begin{cases} y' = 3xy^{1/3} \\ y(-1) = -1 \end{cases}$$

One solves this ODE by the separation of variable method

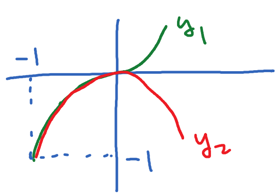
$$\frac{y'}{y^{1/3}} = 3x \rightsquigarrow \frac{dy}{y^{1/3}} = 3x\,dx \rightsquigarrow \int y^{-1/3}\,dy = \int 3x\,dx$$

$$\rightsquigarrow \frac{3}{2} y^{2/3} = \frac{3}{2} x^2 + C$$

Plug $x=-1$ and $y=-1$ to get C. $C=0$.

$$\rightsquigarrow y^{2/3} = x^2 \rightsquigarrow y^2 = x^6 \rightsquigarrow y = \pm x^3$$

We see that $y_1 = x^3$ is a solution. Another solution is $y_2 = \begin{cases} x^3 & \text{if } x<0 \\ -x^3 & \text{if } x \geq 0 \end{cases}$



The ODE with the initial condition has at least two solutions.

Some ODE don't even have a solution, for example

$$xy' + y = 0, \quad y(0) = 1.$$

Differential equations are used to model physical phenomena (such as the falling object, population growth, Newton's Law of Cooling, carbon dating).

If the differential equation doesn't have a solution or if it has more than one solution then it loses the predicting power.

Question: In what case do we know if an ODE has a unique solution?

The answer highly depends on the kind of ODE under concern.

Let's consider ODEs of the form

$$y' = f(x,y) \quad , \quad y(x_0) = y_0$$

Ex
$$y' = xy$$
$$y' = x + y$$
$$y' = \sin(x) + \cos(y)$$
$$y' = e^x y$$
$$\dots$$

Theorem: If f and $\frac{\partial f}{\partial y}$ are continuous on a rectangle $R = (a,b) \times (c,d)$ and $(x_0, y_0) \in (a,b) \times (c,d)$ then there exists an $\varepsilon > 0$ such that the ODE $y' = f(x,y)$, $y(x_0) = y_0$ has a unique solution $y = y(x)$ on $(x_0 - \varepsilon, x_0 + \varepsilon)$.

What is $\frac{\partial f}{\partial y}$? It is the partial derivative of f with respect to y.

Ex

$$f(x,y) = \sin x + \cos y \rightsquigarrow \frac{\partial f}{\partial y} = -\sin y$$

$$f(x,y) = \sin(xy) \rightsquigarrow \frac{\partial f}{\partial y} = x\cos(xy)$$

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