Quiz 1 (Explained)

1) $y' + 2y = 1$, $y(0) = 2$, Solve.

$u = e^{\int 2\,dx} = e^{2x}$

$y = \frac{1}{e^{2x}} \int e^{2x}(1)\,dx$

$y = \frac{1}{e^{2x}}\left(\frac{1}{2}e^{2x} + c\right)$

$y = \frac{1}{2} + Ce^{-2x}$ @ $y(0) = 2$

$2 = \frac{1}{2} + ce^{-2(0)} \Rightarrow C = \frac{3}{2}$

$y = \frac{1}{2} + \frac{3}{2}e^{-2x}$

2) $y' = \frac{x}{y}$, $y(1) = -2$. solve

$yy' = x$

$y\,dy = x\,dx \Rightarrow \int y\,dy = \int x\,dx$

$\frac{1}{2}y^2 = \frac{1}{2}x^2$

$y^2 = x^2 + C_1$ @ $y(1) = -2$

$(-2)^2 = 1 + C_1$

$3 = c_1$

$y^2 = x^2 + 3$

$y = \pm\sqrt{x^2+3} \Rightarrow y = -\sqrt{x^2+3}$

Because initial condition is neg.

$x'' + 3x' + 2x = 0$

$x_1 = e^{-t}$

$x_2 = e^{-2t}$

If we do $x_1 + x_2 = e^{-t} + e^{-2t}$ and get another solution.

Because: (The equation is linear)

$$\begin{aligned} & x_1'' + 3x_1' + 2x_1 = 0 \\ + \ & x_2'' + 3x_2' + 2x_2 = 0 \\ \hline & (x_1+x_2)'' + 3(x_1+x_2)' + 2(x_1+x_2) = 0 \end{aligned}$$

So $x_1 + x_2$ satisfies the equation.

If the equation is nonlinear, the sum of two solutions may not be a solution.

Ex) $x' = x^2$

$$\begin{aligned} & x_1' = x_1^2 \\ + \ & x_2' = x_2^2 \\ \hline & (x_1+x_2)' = \underbrace{(x_1^2 + x_2^2) \neq (x_1+x_2)^2}_{\text{So it doesn't work.}} \end{aligned}$$

Because I can't plug in $x_1 + x_2$ for x.

But linear equations like this have infinitly many solutions because I can scale a solution or add any multiple solutions to get a new solution.

$c(x'' + 3x' + 2x) = 0$ if x is a solution

$(cx)'' + 3(cx)' + 2(cx) = 0$

From 2 solutions, we get infinitly many solutions.

- If x_1 & x_2 are solutions, then...
  $cx_1 + c_2x_2$ is also a solution for any constants c_1, c_2
- If x_1 & x_2 are solutions, any linear combination of x_1 and x_2 is also a solution.

- But what if we still miss one? What if there is another solution not of the form $c_1x_1 + c_2x_2$?

Some trouble:

$x_1 = e^{-t}$
$x_2 = e^{-2t}$ } all possible solutions are $c_1 e^{-t} + c_2 e^{-2t}$

But

If we started with:

$x_1 = e^{-t}$
$x_2 = -5e^{-t}$ } This will never give us e^{-2t} as a solution cuz $c_1 e^{-t} + c_2(-5e^{-t}) \neq e^{-2t}$

But, e^{-2t} is a solution!

So we missed one!

So there is no gaurenttee we can generate all solutions using any two solutions.

$x_1 = e^{-t}$
$x_2 = -5e^{-t}$ ← Kinda generated by x_1 ($-5x_1 = x_2$)
↳ Redondent

But

$x_1 = e^{-t}$
$x_2 = e^{-2t}$ ← x_1 can never generate x_2
↳ We have more Info

→ x_1 & x_2 are linearly independent

→ x_1 & x_2 are linearly dependent

Def: Two functions x_1 & x_2 are called linearly independent if:

$c_1 x_1 + c_2 x_2 = 0$ for all t implies $c_1 = c_2 = 0$

(If we add them and get 0 w/ $c_1 \neq c_2 \neq 0$ then x_1 & x_2 are linearly dependent)