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One can use the Wronskin to check if a list of functions are lin ind.

Ex) t, $\sin t$

$$W(t) = \begin{bmatrix} t & \sin t \\ 1 & \cos t \end{bmatrix}$$

$\det(W(t)) = t\cos t - \sin t$

$\det(W(0)) = 0\cos(0) - \sin(0) = 0$

$\det(W(\frac{\pi}{2})) = \frac{\pi}{2}(\cos(0)) - \sin(\frac{\pi}{2})$

$0 - 1 = -1$

So t & $\sin t$ are lin ind.

Ex) $\sin t, \cos t, 1$

$$\begin{array}{l} \\ 1^{st} \text{ der} \rightarrow \\ 2^{nd} \text{ der} \rightarrow \end{array} W(t) = \begin{bmatrix} \sin t & \cos t & 1 \\ \cos t & -\sin t & 0 \\ -\sin t & -\cos t & 0 \end{bmatrix}$$

Review det

$$= 1 \cdot \begin{bmatrix} \cos t & -\sin t \\ -\sin t & -\cos t \end{bmatrix}$$

$\det(W(t)) = -\cos^2 t - \sin^2 t = -1 \quad \forall t \neq 0$

So $\sin t, \cos t, 1$ are lin ind.

Back to ODE's:

$x'' + 3x' + 2x = 0$

$x_1 = e^{-t}$ } any lin combo is also a solution

$x_2 = e^{-2t}$ } $c_1e^{-t} + c_2e^{-2t}$

Question was: Are we missing any solutions?

No! because x_1 & x_2 are lin ind.

$$W(x) = \begin{bmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{bmatrix}$$

$\det(W(t)) = e^{-t}(-2e^{-2t}) - (-e^{-t})(e^{-2t}) = -2e^{-3t} + e^{-3t} = -e^{-3t}$

$W(0) = 1(-2) - (-1)(1) = -2 + 1 = -1 \neq 0$

So e^{-t} & e^{-2t} are lin ind.

Homogeneous because $= 0$

linear homogeneous

<u>Thm</u>: Consider the 2nd order ODE $x'' + p(t)x' + q(t)x = 0$

IF x_1 and x_2 are lin independent solutions of $(*)$ then

<u>all</u> solutions to $(*)$ are of the form $c_1x_1 + c_2x_2$.

Ex) $x'' - 5x' + 6x = 0$ (If $p(t)$ & $q(t)$ are constant, this method always works)

Find 2 solutions: (Find x in the form e^{rt} with r = constant to be determined)

Then find $x', x'', x'' - 5x' + 6x$.

$x = e^{rt}$, $x' = re^{rt}$, $x'' = r^2e^{rt}$

$r^2e^{rt} - 5re^{rt} + 6e^{rt} = 0$

$e^{rt}(r^2 - 5r + 6) = 0$

$e^{rt}(r-3)(r-2) = 0$ $r = 3, 2$, $\Rightarrow$ $\boxed{\begin{matrix} x_1 = e^{2t} \\ x_2 = e^{3t} \end{matrix}}$ ← 2 solutions

$$W(x) = \begin{bmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{bmatrix}$$

$\det(W(x)) = e^{2t}(3e^{3t}) - 2e^{2t}e^{3t}$

$3e^{5t} - 2e^{5t} = e^{5t}$ @ $t = 0$ $e^{5(0)} = 1 \neq 0$

So e^{2t} and e^{3t} are lin ind.

So they provide all the possible solutions

(By 2, get infinite free)

↳ $c_1e^{2t} + c_2e^{3t}$ = All solutions

Characteristic Equation is the "r version of this"

Solve ODE's of the form $ax'' + bx' + cx = 0$:
With a, b, c constants.
Find solutions of the form e^{rt}. (If you find 2 that are lin ind you get them all)
Plug $x = e^{rt}$ and its derivatives into the equation.

$x' = re^{rt}$, $x'' = r^2 e^{rt}$

$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0 \Rightarrow e^{rt}(ar^2 + br + c) = 0$

Solve for r.
Most times this equations will have 2 real roots (that are different)
If:
1) The equation has 2 distinct real roots, r_1 & r_2, then,
$x_1 = e^{r_1 t}$ and $x_2 = e^{r_2 t}$ are two solutions.
These solutions turn out to be lin ind. therefore,
all solutions to the ODE are lin combos of
x_1 and $x_2 \Rightarrow x = c_1 e^{rt} + c_2 e^{rt}$

Ex) Solve the initial value problem $x'' - x' - 6x = 0$, $x(0) = 3$
$x'(0) = -1$

$x = e^{rt}$, $x' = re^{rt}$, $x'' = r^2 e^{rt}$

$r^2 e^{rt} - re^{rt} - 6(e^{rt}) = 0$

$e^{rt}(r^2 - r - 6) = 0$

$(r-3)(r+2) = 0 \quad e^{3t}$ & e^{-2t}

All solutions: $x_1 = e^{3t}$, $x_2 = e^{-2t}$

$x = c_1 e^{3t} + c_2 e^{-2t}$

Now find c_1 & c_2

$x = c_1 e^{3t} + c_2 e^{-2t}$ @ $x(0) = 3$

$x' = 3c_1 e^{3t} - 2c_2 e^{-2t}$ $\quad x'(0) = -1$

$3 = c_1 e^{3(0)} + c_2 e^{-2(0)}$

$-1 = 3c_1 e^{3(0)} - 2c_2 e^{-2(0)}$ (get $c_1 = 1$, $c_2 = 2$)

$x = e^{3t} + 2e^{-2t}$