$a\ddot{x} + b\dot{x} + cx = 0$
↑ 2^{nd} Order Linear ODE with Constant Coefficients.

Characteristic equation:
$ar^2 + br + c = 0$, solve for r.

If we have 2 different real roots we get 2 solutions
- If (*) has two distinct real roots r_1 + r_2, then all solutions are of the form
$x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ ← General Solution to the ODE.

What if (*) has a double root? ($r_1 = r_2$)
If r is a double root then we get one solution,
$x_1 = e^{rt}$. (Now we have to find another solution)
Strategy: Seek x_2 of the form $x_2 = ye^{rt}$ where $y(t) = y$
Note: y is an unknown function that is not a constant.
$x_2 = ye^{rt}$, $x_2' = y'e^{rt} + rye^{rt}$, $y'' = y''e^{rt} + y're^{rt} + ry'e^{rt} + r^2ye^{rt}$
$= y''e^{rt} + 2ry'e^{rt} + r^2ye^{rt}$

$a \cdot (x_2'' = e^{rt}(y'' + 2ry' + r^2y))$
$b \cdot (x_2' = e^{rt}(y' + ry))$
$+ \; c \cdot (x_2 = ye^{rt})$
$e^{rt}((ay'' + by' + cy) + (2ary' + bry) + ar^2y) = 0$
Lets group differently
$cy + bry + ar^2y = y(c + br + ar^2) = y(0) = 0$
So I can cancel these terms
$(ay'' + by' + 2ary')e^{t} = 0$

"Sometimes we wonder why life treats us so well"

→ $by' + 2ray' = (b + 2ra)y' = 0(y')$ because $r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{b}{2a}$
$b + 2ra = c$
$= ay''e^{rt} = 0$ so $ay'' = 0$ but $a \neq 0$ so $y'' = 0$.

The method of finding x_2 from the form $x_2 = yx_1$ is called the variation of constant.
Instead of cx, we do yx.
And y cant be constant and $y'' = 0$ so $y = t$ (choose simple)
So $x_2 = ate^{rt}$
The two solutions of the ODE are $x_1 = e^{rt}$, $x_2 = te^{rt}$

Ex) $x'' + 4x' + 4x = 0$, $x(0) = 1$, $x'(0) = -2$

Characteristic eq: $r^2 + 4r + 4 = 0$

$(r+2)^2$ $r = -2, -2$

$x_1 = e^{-2t}$

$x_2 = te^{-2t}$ ← Every time w/ double Roots!

All solutions are of the form $x = c_1 e^{-2t} + c_2 t e^{-2t}$

$x' = -2c_1 e^{-2t} + Ce^{-2t} + c_2 t(-2) e^{-2t}$

@ $x(0)$ $c_1 \rightsquigarrow c_1 = 1$

$x'(0) = -2c_1 + c_2 \rightsquigarrow c_2 = 0$

So $x = e^{-2t}$