$ax'' + bx' + cx = 0$

<u>3 Cases</u>: $ar^2 + br + c = 0$

- $\to$ 2 distinct roots
- $\to$ 1 double root

} could be on exam

- $\to$ 2 complex roots

If 2 distinct roots, r_1 & $r_2 \Rightarrow x_1 = e^{r_1 t}$ $\quad x_2 = e^{r_2 t}$

So $x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

If 1 double root $r \Rightarrow x_1 = e^{rt}$ $\quad x_2 = te^{rt}$

So $x = (c_1 + tc_2)e^{rt} = c_1 e^{rt} + c_2 t e^{rt}$

If 2 complex roots, $\alpha \pm i\beta$ (always in pairs)

$\Rightarrow x_1 = e^{\alpha t}\cos(\beta t),\ x_2 = e^{\alpha t}\sin(\beta t)$

So $x = (c_1\cos(\beta t) + c_2\sin(\beta t))e^{\alpha t}$

$= c_1\cos(\beta t)e^{\alpha t} + c_2\sin(\beta t)e^{\alpha t}$

Why is it that $x_1 = e^{\alpha t}\cos(\beta t)$, $x_2 = e^{\alpha t}\sin(\beta t)$

Recall if r satisfies $ar^2 + br + c = 0$ then $x = e^{rt}$ solves the ODE

In the case $r = \alpha + i\beta$:

$x = e^{(\alpha + \beta i)t} = e^{\alpha t + i\beta t} = e^{\alpha t}e^{i\beta t}$

Euler's formula: $e^{i\theta}$ with $\theta \in \mathbb{R}$, $= \cos\theta + i\sin\theta$

Side comment: $e^{i\pi} = \cos\pi + i\sin\pi$

$e^{i\pi} = -1 \Rightarrow e^{i\pi} + 1 = 0$ ← Contains all the important constants of math.

Why does Euler's formula work?

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

So $e^{ix} = 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \dots = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right) + \left(ix - i\frac{x^3}{3!} + i\frac{x^5}{5!} + \dots\right)$

So $e^{ix} = \cos x + i\sin x$

$x = e^{\alpha t}e^{i\beta t} \Rightarrow e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))$

$\tilde{x} = e^{(\alpha - i\beta)t} = e^{\alpha t}e^{-i\beta t} = e^{\alpha t}(\cos(-\beta t) + i\sin(-\beta t))$

$= e^{\alpha t}(\cos(\beta t) - i\sin(\beta t))$

<u>Solutions!</u>

$\frac{x+x^2}{2} = e^{\alpha t}\cos\beta t = x_1$

$\frac{x-x^2}{2i} = e^{\alpha t}\sin\beta t = x_2$

All solutions will be a linear combination of x_1 & x_2

Check if x_1 & x_2 are lin ind!!
(Using Wronskines)

Ex) Solve the initial value problem

$x'' + 4x' + 5x = 0$

$x(0) = -1$

$x'(0) = 2$

$r^2 + 4r + 5 = 0$

$\frac{-4 \pm \sqrt{4^2 - 4(5)}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = -2 \pm i$

So $r = -2 \pm i$

$x_1 = e^{-2t}\cos(t)$, $x_2 = e^{-2t}\sin(t)$

$x = e^{-2t}\cos(t) + e^{-2t}\sin(t)$

$x' = 2e^{-2t}\cos t - e^{-2t}\sin t + 2e^{-2t}\sin t + e^{-2t}\cos t$

$x' = 3e^{-2t}\cos t + e^{-2t}\sin t$

@ $x(0) = -1$ & $x'(0) = 2$

$x = c_1 e^{-2t}\cos(t) + c_2 e^{-2t}\sin(t)$

$x' = 3c_1 e^{-2t}$

Needs redone but close

Should get $\alpha = -2$, $\beta = 1$ (✓)

$x = (c_1\cos t + c_2\sin t)e^{-2t} \Rightarrow x(0) = c_1 \Rightarrow c_1 = -1$

$x' = (-c_1\sin t + c_2\cos t)e^{-2t} + (c_1\cos t + c_2\sin t)(-2e^{-2t})$

@ $x'(0) = 2 \Rightarrow c_2 = 0$

$x = -\cos t\, e^{-2t}$