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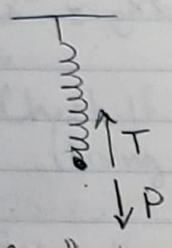
We know how to solve $ax''+bx'+c=0$
But what if its not 0?



$mx''+kx=0$
Without Friction

With friction:
$mx''+\mu x'+kx=0$

But what if we hold the spring at the top and lifted up and down as it stretched?
Thats an external force!

$mx''+\mu x'+kx=F$ ← External Force

Now we have a nonhomogeneous 2^{nd} order ODE.
How to solve? (Depends on F but...)
3 ~~Two~~ steps:

1) Solve the problem as $mx''+\mu x'+kx=0$ (x_c) ← Complementary Solution
2) Find one particular solution to $mx''+\mu x'+kx=F$ (x_p) ← Particular Solution (Particular Integral in the book)
   - Only 1!
3) Combine: All solutions to the equation $mx''+\mu x'+kx=F$ are $x=x_c+x_p$

Why does this work?

$$mx''+\mu x'+kx=F \quad \text{with } x \text{ being some solution}$$
$$-\left(mx_p''+\mu x_p'+kx_p=F\right) \quad \text{with } x_p \text{ being some solution}$$
$$m(x-x_p)''+\mu(x-x_p)'+k(x-x_p)=0$$

So $x-x_p$ is a solution!

$u=x-x_p \Rightarrow mu''+\mu u'+ku=0$

And we now how to get u. (Its x_c from earlier)

So $x=\underset{x_c}{u}+x_p$

Ex) Find all solutions to the ODE

$x'' + 3x' + 2x = 4t$

Step 1) $x'' + 3x' + 2x = 0 \Rightarrow r^2 + 3r + 2 = 0 \quad r = -1, -2$

Solve the complementary (homogeneous) equation

$x_1 = e^{-t}, \quad x_2 = e^{-2t}$

$x_c = c_1 e^{-t} + c_2 e^{-2t}$

Step 2) Find a particular solution to $x'' + 3x' + 2x = 4t$

(Guess)

What x makes this solution 4t?

$x \overset{?}{=} 2t \quad 0 + 3(2) + 2(2t) = 6 + 4t$

$x \overset{?}{=} 2t - 3 \quad 0 + 3(2) + 2(2t-3) = 6 + 4t - 6 = 4t$ ✓

So $x_p = 2t - 3$ (By guessing)

Step 3) Combine as $x = x_c + x_p$

$x = c_1 e^{-t} + c_2 e^{-2t} + 2t - 3$

All possible solutions

Ex) Solve $x'' + 3x' + 2x = 4t$ with the initial conditions $x(0) = 1, x'(0) = 1$

$x = c_1 e^{-t} + c_2 e^{-2t} + 2t - 3$

$x' = -c_1 e^{-t} - 2c_2 e^{-2t} + 2$

$\Downarrow$

$1 = c_1 e^0 + c_2 e^0 + 2(0) - 3 \Rightarrow 1 = c_1 + c_2 - 3$

$1 = -c_1 e^0 - 2c_2 e^0 + 2 \qquad + \; 1 = -c_1 - 2c_2 + 2$

$2 = 0 - 1c_2 - 1$

$2 = -1c_2 - 1$

$-3 = c_2$

$1 = c_1 - 3 - 3$

$1 = c_1 - 6$

$7 = c_1$

$\boxed{x = 7e^{-t} - 3e^{-2t} + 2t - 3}$

Ex) Find a particular solution to $x'' + 2x' + 3x = t^2$
(Guessing again)

$x \stackrel{?}{=} -t^2 \Rightarrow x' = -2t \quad x'' = -2$

$-2 + 2(-2t) + 3(-t^2)$

$x = t^2 \Rightarrow x' = 2t \quad x'' = 2$

$2 + 2(2t) + 3t^2 = 3t^2 + 4t + 2$

$x = \frac{1}{3}t^2 \Rightarrow x' = \frac{2}{3}t, \quad x'' = \frac{2}{3}$

$\frac{2}{3} + 2(\frac{2}{3}t) + 3(\frac{1}{3}t^2) = t^2 + \frac{4}{3}t + \frac{2}{3}$

What about $x = At^2 + Bt$

$x' = 2At + B, \quad x'' = 2A$

$2A + 2(2At + B) + 3(At^2 + Bt)$

$3At^2 + (3B + 4A)t + (2B + 2A) = t^2$

$3A = 1$ cuz $3At^2 = t^2$

$A = \frac{1}{3}$

And $3B + 4A = 0$ cuz $(3B + 4A)t = 0t$

$3B + \frac{4}{3} = 0$

$3B = \frac{-4}{3}$

$B = \frac{-4}{9}$

And $2B + 2A = 0$ cuz the constant on the right is 0

But $2(\frac{-4}{9}) + 2(\frac{1}{3}) \neq 0$

What if we have 3 unknowns?

$x = At^2 + Bt + C, \quad x' = 2At + B \quad x'' = 2A$

$2A + 2(2At + B) + 3(At^2 + Bt + C) = t^2$

$3At^2 + (3B + 4A)t + (2B + 2A + 3C) = t^2$

$A = \frac{1}{3}$

$B = \frac{-4}{9}$

$C = \frac{2}{27}$ ← Tuan solved on board

$= 0$