Consider ODE's of the form

$$ax'' + bx' + cx = f(t)$$

(a, b, c: constants)

General Solution: $x = x_c + x_p$

(x_c: Complimentary Solution; x_p: Particular Solution)

x_c comes from the solution of $ax'' + bx' + cx = 0$

x_p is a particular solution of $ax'' + bx' + cx = f(t)$

Recall:

$x'' + 3x' + 2x = 4t$, gives $x_p = 2t - 3$ ($At + B$)

$x'' + 2x' + 3x = t$ guess $x_p = At^2 + Bt + C$

Observation: If the right side is a poly of degree n then the solution is of degree n.

More formally:

If $f(t)$ is a polynomial of degree n then we guess $x_p = A_n t^n + \ldots + A_1 t + A_0$ and the solve for $A_n, \ldots, A_1, A_0$.

Ex) Find a particular solution to $x'' - 5x' + 2x = 3t$

So $x_p = At + B$

$$x = At + B$$
$$x' = A$$
$$x'' = 0$$

$$0 - 5(A) + 2(At + B) = 3t$$
$$-5A + 2At + 2B = 3t + 0$$

$2At = 3t$ $-5A + 2B = 0$

$2A = 3$ $-\frac{15}{2} + 2B = 0$

$A = \frac{3}{2}$ $2B = \frac{15}{2}$ $\Rightarrow$ $\boxed{x_p = \frac{3}{2}t + \frac{15}{4}}$

Check: $x = \frac{3}{2}t + \frac{15}{4}$ $x' = \frac{3}{4}$ $x'' = 0$

$$-5\left(\frac{3}{2}\right) + 2\left(\frac{3}{2}t + \frac{15}{4}\right) = 3t$$

$$\cancel{-\frac{15}{2}} + 3t + \cancel{\frac{15}{2}} = 3t \Rightarrow 3t = 3t$$

Ex) Find a particular solution to $x'' - 5x' = 3t$

So x' has to give me t (or t^2)

$x_p = At^2 + Bt + C$

$x' = 2At + B$

$x'' = 2A$

$2A - 5(2At + B) = 3t$

$2A - 10At - 5B = 3t$

$-10At = 3t$ $\quad$ $2A - 5B = 0$ $\quad$ $C = 0$

$-10A = 3$ $\quad$ $2(\frac{-3}{10}) - 5B = 0$

$A = \frac{-3}{10}$ $\quad$ $\frac{-3}{5} - 5B = 0$

$-5B = \frac{-3}{5}$

$B = \frac{-3}{25}$

$x_p = \frac{-3}{10}t^2 - \frac{3}{25}t$

$x' = \frac{-6}{10}t - \frac{3}{25}$ $\quad$ $x'' = \frac{-6}{10}$

$\frac{-6}{10} - 5(\frac{-3}{5}t - \frac{3}{25}) = 3t$

$\cancel{\frac{-3}{5}} + 3t + \cancel{\frac{3}{5}} = 3t$ so $3t = 3t$ ✓

guess $x = At + B$

$x' = A$ $\quad$ $x'' = 0$

So $0 - 5A = 3t$

Why does B disappear?

$x_c'' - 5x_c' = 0$

$r^2 - 5r = 0$ $\quad$ $r = 0, 5$

$x_c = c_1 e^{0t} + c_2 e^{5t} = \underline{c_1 + c_2 e^{5t}}$

This means any constant in x_p is already accounted for, not needed again.
So B is not needed (disappears)
B is part of x_c because of c_1

By raising the degree of the guess of x_p we get more freedom to play with.

$ax'' + bx' + cx = f t$

If $f(t)$ is an n^{th} degree polynomial...

First guess for x_p is $x_p = A_n t^n + \dots + A_1 t + A_0$

If any of these terms $(A_n t^n, \dots, A_1 t, A_0)$ appear in x_c then raise the degree of the guess.

↳ Same as multiplying the first guess by t.

Ex) $At + B \Rightarrow At^2 + Bt + C$ but $C \in x_c$ so $At^2 + Bt$ is the best guess which is $t(At + B)$

What if $f(t)$ is not a polynomial?

Ex) $x'' - 3x' + 2x = 2e^{-t}$

What should we guess?

$x = Ae^{-t}$ $x' = -Ae^{-t}$ $x'' = Ae^{-t}$

$Ae^{-t} - 3(-Ae^{-t}) + 2(Ae^{-t}) = 2e^{-t}$

$Ae^{-t} + 3Ae^{-t} + 2Ae^{-t} = 2e^{-t}$

$6Ae^{-t} = 2e^{-t}$

$A = \frac{1}{3}$ so $x_p = \frac{1}{3}e^{-t}$

If $f(t)$ is of the form $\underbrace{P(t)}_{\text{Poly w/ deg } n} e^{at}$

1^{st} guess is $x_p = (A_n t^n + \dots + A_1 t + A_0)e^{at}$

(If it doesn't work / appear in x_c multiply by t)