What happens if the natural frequency is equal to the forcing frequency?

Ex) $\overset{a}{x''} + \overset{m}{4}\overset{k}{x} = 2\cos(\overset{w}{2t})$, $m=1$ $k=4$ $w=2$ $a=2$

$x_c = c_1 \cos w_0 t + c_2 \sin w_0 t$

$x_c = c_1 \cos 2t + c_2 \sin 2t$

$x_p = At\cos(2t) + Bt\sin(2t)$

$x_p' = A\cos(2t) + 2At\cos(2t) + B\sin(2t) + 2Bt\sin(2t)$

$x_p'' = 2A\cos(2t) + 2A\cos(2t) + 4At\cos(2t) + 2B\sin(2t) + 2B\sin(2t) + 4Bt\sin(2t)$

$= 4A\cos(2t) + 4At\cos(2t) + 4B\sin(2t) + 4Bt\sin(2t)$

5/19 $x'' + 4x = \cos(2t)$

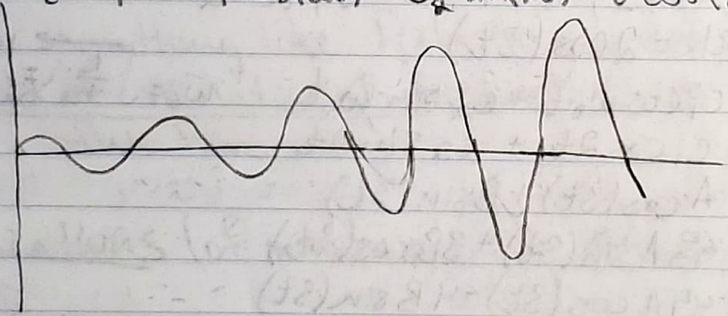
$x_p = At\cos(2t) + Bt\sin(2t)$

Lets assume $A=1, B=0$ (lots of work)

$x_p = t\cos(2t)$

General Sol.

$x = x_c + x_p = c_1\cos(2t) + c_2\sin(2t) + t\cos(2t)$



This is if the natural frequency is equal to the forcing frequency.

What happens if we add friction?

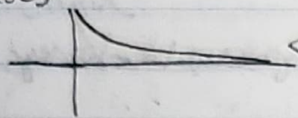
With Friction

$mx'' + \mu x' + kx = F(t)$

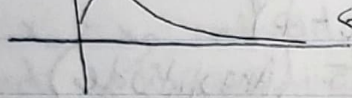
← Still a periodic force

If $F(t)=0$

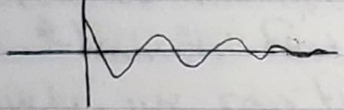
3 cases:



← Overdamped
(Too much friction) ← 2 real roots



← Critically damped
(Friction but still vibration) ← Double root



← Under damped
(Not enough Friction) ← 2 complex roots

Question: Will an over damped spring respond to being moved periodically? What happens to the amplitude / frequency?

Ex) $m=1$, $\mu=3$, $k=2$, $F(t)=\cos(t)$

$x'' + 3x' + 2x = \cos t$

$x'' + 3x' + 2x = 0$ $\quad r^2+3r+2=0 \quad r=-1,-2$ ← Overdamped

$x_c = c_1e^{-t} + c_2e^{-2t}$

$x_p = A\cos t + B\sin t$, $x' = -A\sin t + B\cos t$, $x'' = -A\cos t - B\sin t$

$-A\cos t - B\sin t - 3A\sin t + 3B\cos t + 2A\cos t + 2B\sin t = \cos t$

$-A + 3B + 2A = 1 \qquad -B - 3A + 2B = 0 \Rightarrow \qquad 3B + A = 1$

$3B + A = 1 \qquad\qquad B - 3A = 0 \qquad\qquad - \underline{3B - 9A = 0}$

$B - \frac{3}{10} = 0 \qquad\qquad 10A = 1 \quad A = \frac{1}{10}$

$B = \frac{3}{10}$

$x = c_1e^{-t} + c_2e^{-2t} + \frac{1}{10}\cos t + \frac{3}{10}\sin t$

← If c_1 & c_2 are 0 then x is periodic.
If not, x is not periodic

If $x(0) = \frac{1}{10}$ & $x'(0) = \frac{3}{10}$ then this system is periodic
But what if t is very large? (you wait a long time)
then e^{-t} & e^{-2t} are very small / don't do much. So the periodic part is more

$$x = \underbrace{c_1 e^{-t} + c_2 e^{-2t}}_{\text{Transient Solution}} + \underbrace{\tfrac{1}{10}\cos t + \tfrac{3}{10}\sin t}_{\text{Steady-state Solution}}$$

Most strong if t is small, eventually it dies out.

So this function is periodic if you wait long enough.

What is the amplitude/frequency?

$$\tfrac{1}{10}\cos t + \tfrac{3}{10}\cos t = A\sin(t-\phi)$$

$$A = \sqrt{\left(\tfrac{1}{10}\right)^2 + \left(\tfrac{3}{10}\right)^2} = \sqrt{\tfrac{1+9}{100}} = \tfrac{1}{\sqrt{10}} \text{ (Amplitude)}$$

Recall: $\cos(t)\cos(\phi) + \sin(t)\sin(\phi)$

$$\phi = \frac{1/10}{A} = \frac{1}{\sqrt{10}}$$

$$\sin(\phi) = \frac{3/10}{A} = \frac{3}{\sqrt{10}}$$

$$\underbrace{A}_{\text{Amplitude}}\ \underbrace{\sin(t-\phi)}_{\text{Phase shift}}$$

The amplitude of $F(t) = \cos t$ is 1

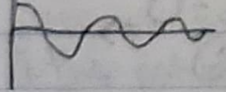
So the amplitude of a spring in honey is smaller.

The frequency is the same (1)

from $\cos(1t)$ & $A\cos(1t-\phi)$

A lot of time in real life, μ is quite small in

$mx'' + \mu x' + kx = F(t)$ (so small friction) $\rightarrow$ (underdamped by nature)

 (like the shocks in a car hitting a pothole

Consider $F(t) = a\cos(\omega t)$ (different ω gives different frequencies

The solution vibrates the most (largest amplitude)

when $\omega = \sqrt{\frac{k}{m}} \cdot \sqrt{1 - \frac{\mu^2}{2mk}}$ (Resonance)

This is what happens in real life, because the is always friction.

(Think about a swing & pushing someone)

5/22 Solve equations of the form $ax'' + bx' + cx = f(t)$

- Find x_c & x_p and add
  - find x_p through the method of undetermined coefficients. (for certian types of $f(t)$)

    learn this first to learn what the heck we doing

Alternatively, we can find a particular solution using the Method of Variation of Parameters

Suppose $x_c = c_1x_1 + c_2x_2$

Complementary Solution

If I plug this into $ax'' + bx' + cx = f(t)$

we get $ax_c'' + bx_c' + cx_c = 0$ $\forall c_1, c_2$, c_1, c_2 are constants

Idea: Allow c_1, c_2 to be functions of t (rather than constants)

We'll find c_1, c_2 so that the right hand side is equal to f.

For ease, lets rename c_1 as u_1 and c_2 as u_2

- u_1, u_2 are functions of t (not constants)

We are seeking for a particular solution of the form $x = u_1x_1 + u_2x_2$, where x_1, x_2 are linearly independent solutions of the homogeneous equation.

We need to determine u_1, u_2