$ax'' + bx' + cx = f$

$x = u_1 x_1 + u_2 x_2$

x_1, x_2 are lin ind. solutions to the homogeneous eq.

$u_1' = \frac{x_2 g}{W}$ $\quad g = \frac{f}{a}$

$u_2' = -\frac{x g}{W}$ $\quad W = \begin{bmatrix} x_1 & x_2 \\ x_1' & x_2' \end{bmatrix} = x_1 x_2' - x_2 x_1'$

Note: a, b, c don't have to be constants.
(They can be functions of t, but its easier if a, b, c are constant)

Ex) $x'' - 2x' - 3x = te^t$

Find general solution to this ODE. (2 possible methods → Undetermined Coefficients / → Variation of parameters

First way

$x'' - 2x' - 3x = 0$

$r^2 - 2r - 3 = 0 \Rightarrow (r-3)(r+1) = 0,\ r = -1, 3$

$x_c = c_1 e^{-t} + c_2 e^{3t}$

$x_p = At + B(e^t) = Ate^t + Be^t$

$x_p' = Ae^t + Ate^t + Be^t$

$x_p'' = Ae^t + Ae^t + Ate^t + Be^t = 2Ae^t + Ate^t + Be^t$

$-2Ae^t - 2Ate^t - 2Be^t$

$-3Ate^t - 3Be^t$

$-4Ate^t - 4Be^t$

$-4Ate^t - 4Be^t = te^t$

$-4A = 1 \qquad B = 0$

$A = -\frac{1}{4}$

$x = c_1 e^{-t} + c_2 e^{3t} - \frac{1}{4} te^t$

Ex cont.) $x'' - 2x' - 3x = te^t$

$x_1 = e^{3t} \quad x_2 = e^{-t}$

$W = \begin{bmatrix} e^{3t} & e^{-t} \\ 3e^{3t} & -e^{-t} \end{bmatrix} = e^{3t}(-e^{-t}) - (e^{-t})(3e^{3t})$

$-e^{2t} - 3e^{2t} = -4e^{2t}$

$g = \frac{te^t}{1} = te^t$

$u_1' = \frac{-e^{-t}(te^t)}{-4e^{2t}} = \frac{t}{4e^{2t}} \qquad u_2' = \frac{e^{3t}(te^t)}{-4e^{2t}} = \frac{te^{4t}}{4e^{2t}} = \frac{-te^{2t}}{4}$

$u_1 = \frac{-1}{4}\int te^{-2t}\,dt$

$u = t \qquad v = \frac{-1}{2}e^{-2t}$

$du = 1dt \quad dv = e^{-2t}dt$

$\frac{1}{4}\left(\frac{-1}{2}te^{-2t} - \int \frac{-1}{2}e^{-2t}\,dt\right)$

$\frac{1}{4}\left(-\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t}\right)$

$u_1 = \frac{-1}{8}te^{-2t} - \frac{1}{16}e^{-2t} + c_1$

$u_2 = \frac{1}{4}\int te^{2t}\,dt$

$u = t \qquad v = \frac{1}{2}e^{2t}$

$du = 1dt \quad dv = e^{2t}dt$

$\frac{-1}{4}\left(\frac{1}{2}te^{2t} - \int \frac{1}{2}e^{2t}dt\right)$

$-\frac{1}{4}\left(\frac{1}{2}te^{2t} - \frac{1}{4}e^{2t}\right) + C_2$

$u_2 = \frac{-1}{8}te^{2t} + \frac{1}{16}e^{2t} + C_2$

$x = \left(\frac{-1}{8}te^{-2t} - \frac{1}{16}e^{-2t} + c_1\right)e^{3t} + \left(\frac{-1}{8}te^{2t} + \frac{1}{16}e^{2t} + C_2\right)e^{-t}$

$-\frac{1}{8}te^t \cancel{- \frac{1}{16}e^t} + c_1e^{3t} - \frac{1}{8}te^t + \cancel{\frac{1}{16}e^t} + c_2e^{-t}$

✓ $X = -\frac{1}{4}te^t + c_1e^{3t} + c_2e^{-t}$

Remember! Integration by parts: $\int u\,dv = uv - \int v\,du$
(Try to pick u so du goes away/is constant)