Lecture 3

Thursday, April 6, 2023 10:14 PM

* Questions ....

Autonomous 1st order ODE: $x' = f(x)$

Ex $x' = x$

$$\rightsquigarrow \frac{x'}{x} = 1 \rightsquigarrow \frac{dx}{x} = dt \rightsquigarrow \int \frac{dx}{x} = \int dt$$

$$\rightsquigarrow \ln|x| = t + C$$

$$\rightsquigarrow |x| = e^{t+C}$$

There are infinitely many solutions (one for each C). To get a unique solution, we need an initial condition.

Say, $x(0) = 5$. Then $C = \ln 5$ and

$$\boxed{x = 5e^t}$$

Ex Solve the ODE $x' = x^2$ with the initial condition $x(0) = 1$.

$$\frac{x'}{x^2} = 1 \rightsquigarrow \frac{dx}{x^2} = dt \rightsquigarrow \int \frac{dx}{x^2} = \int dt$$

$$\rightsquigarrow -\frac{1}{x} = t + C$$

At $t = 0$: $-1 = 0 + C \rightsquigarrow C = -1$

Therefore, $x = \frac{1}{1-t}$.

Ex $\quad x' = x^2 + x, \quad x(0) = 1$

$\rightsquigarrow \frac{x'}{x^2+x} = 1 \rightsquigarrow \int \frac{dx}{x^2+x} = \int dt \qquad (*)$

We will use partial fraction decomposition to find the integral

$$\int \frac{dx}{x^2+x}$$

Note that

$$\frac{1}{x^2+x} = \frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

$\rightsquigarrow 1 = A(x+1) + Bx$

At $x=0 \rightsquigarrow A = 1$

At $x=-1 \rightsquigarrow B = -1$

$$\frac{1}{x^2+x} = \frac{1}{x} - \frac{1}{x+1}$$

Thus, $\int \frac{dx}{x^2+x} = \int \frac{dx}{x} - \int \frac{dx}{x+1} = \ln|x| - \ln|x+1| + C$

Because $x(0) = 1 > 0$, we can drop the absolute value signs.

(*) now becomes $\ln x - \ln(x+1) = t + C$

At $t=0$: $\ln 1 - \ln 2 = 0 + C \rightsquigarrow C = -\ln 2$.

Thus, $\ln x - \ln(x+1) = t - \ln 2$

$$\rightsquigarrow \ln \frac{x}{x+1} = t - \ln 2$$

$$\rightsquigarrow \frac{x}{x+1} = e^{t-\ln 2} = \frac{1}{2} e^t$$

$$\rightsquigarrow x = \frac{1}{2} e^t (x+1) = \frac{1}{2} e^t x + \frac{1}{2} e^t$$

$$\rightsquigarrow x = \frac{\frac{1}{2} e^t}{1 - \frac{1}{2} e^t}$$

We see that with simple function $f(x) = x^2 + x$, the solution is quite complicated and the process of getting the solution is nontrivial. For slightly more complicated f, perhaps we won't be able to get an explicit formula for the solution. We may have to settle on a less ambitious goal.

Next time, we will learn some properties of the solution of an autonomous ODE without solving for it.