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Power series

$f(x) = a_0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + \ldots = \sum_{n=0}^{\infty} a_n x^n$

This is a power series representation of $f(x)$.

How to get $a_0, a_1, a_2, \ldots$ from f

$f(0) = a_0$

$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots$

$f'(0) = a_1$

$f''(x) = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \ldots$

$f''(0) = 2a_2 \Rightarrow \frac{1}{2} f''(0) = a_2$

So

$a_3 = \frac{1}{6} f'''(0)$

In general: $a_n = \frac{1}{n!} f^{(n)}(0)$

The series $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ is called the Taylor Series of f.
(OR the McClaurin Series)

Some famous power series...

$e^x = f(x)$

$a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!} \Rightarrow e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Note: $e = e^1 = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$

Ex) $y'' + xy = e^x$, $\boxed{y(0) = 1,\ y'(0) = 2}$

Write y as a power series

$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \Rightarrow$ Find all coeff. to find the sol.

$\boxed{a_0 = 1,\ a_1 = 2}$, $a_2 = \frac{y''(0)}{2}$ what is $y''(0)$?

@ $x = 0 \Rightarrow y'' + 0y = e^0 = y''(0) = 1 \Rightarrow \boxed{\frac{1}{2} = a_2}$

Method 1

a_3? $\frac{y'''(0)}{6} = ?$

$y'' + xy = e^x$

$y''' + 1y + xy' = e^x$

$y'''(0) + 1y(0) + x(y''(0)) = e^0$

$y'''(0) + 1(1) \quad 0 \quad = 1$

$\boxed{y'''(0) = 0}$ $\boxed{a_3 = 0}$

$y'' + xy = e^x$

$y''' + 1y + xy' = e^x$

$y'''' + y' + y' + xy'' = e^x$

$y'''' + 2 + 2 = 1$

$y'''' = -3$

$a_4 = \frac{-3}{24} = \frac{-1}{8}$

$\cancel{y'''' + y' + 1y'' + xy'''} = \cancel{e^x}$

$y = 1 + 2x + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \ldots$

Ex) $y'' + xy = e^x$ (Method 2)

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$xy = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n$$

(Shifting)

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

(First term is 0 so we can ignore it)

$$y'' = \sum_{n=1}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

Shifting up 2 (Everything goes up 2, n under Σ goes down 2)

Plug them into $y'' + xy = e^x$ and see