From Last time: $y'' + xy = e^x$

$\therefore\ y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots = \sum_{n=0}^{\infty} a_n x^n$

$y' = a_1 + 2a_2 x + 3a_3 x^2 + \dots = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} (n+1) a_{n+1} x^n$

$$y'' = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$$

$$xy = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n$$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

Now put it together...

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

Notice n=0 has no pair in x but from n=1 up it has a pair.

$n=0$ $\sum_{n=0} (0+1)(0+2) a_{0+2} x^0 = a_2$

$$a_2 + \sum_{n=1}^{\infty} \left((n+1)(n+2) a_{n+2} + a_{n-1}\right) x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

Have to single out n=0 here too

$$\sum_{n=0} \frac{1}{0!} x^0 = 1$$

$$a_2 + \sum_{n=1}^{\infty} \left((n+1)(n+2) a_{n+2} + a_{n-1}\right) x^n = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} x^n$$

So

$a_2 = 1$ $\qquad \sum_{n=1}^{\infty} \left((n+1)(n+2) a_{n+2} + a_{n-1}\right) x^n = \sum_{n=1}^{\infty} \frac{1}{n!} x^n$

$\left((n+1)(n+2) a_{n+2} + a_{n-1}\right) x^n = \frac{1}{n!} x^n$ ← Must be true for all n

what if $n=1$

$6a_3 + a_0 = 1$ (We should know $a_0$ to solve this) $y(0)=x$ $\quad y'(0)=x$

Differentiating was a half life ago

What if $n=2 \Rightarrow 12a_4 + a_1 = \frac{1}{2}$ (We should know $a_1$ to solve)

$n=3 \Rightarrow$ We have $a_2$

$n=4 \Rightarrow$ We found $a_3$

and we can solve for all n, one n at a time.

R

This is a recursive formula.

$$\hookrightarrow a_{n+2} = \frac{\frac{1}{n!} - a_{n-1}}{(n+1)(n+2)}$$

Ex) Find the general solution to
$y' = (x+1)y$ using power series.

$$y = a_0 + a_1 x^1 + a_2 x^2 + \dots = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \dots = \sum_{n=0}^{\infty} n a_n x^{n-1}$$
$$= \sum_{n=1} (n+1) a_{n+1} x^n$$

$$(x+1)y = (x+1)\sum_{n=0}^{\infty} a_n x^n$$
$$= \sum_{n=0} a_n x^{n+1} + a_n x^n$$

$$\sum_{n=0}^{\infty} a_n x^{n+1} + a_n x^n = \sum_{n=1}^{\infty} (n+1) a_{n+1} x^n$$

$$\sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{x=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} (n+1) a_{n+1} x^n$$

$$\sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} (n+1) a_{n+1} x^n$$

$$(a_{n-1} + a_n) x^n = (n+1) a_{n+1} x^n$$
$$a_{n-1} + a_n = (n+1) a_{n+1}$$
$$a_n = (n+1) a_{n+1} - a_{n-1}$$

$$xy + y = a_0 + \sum_{n=1}^{\infty} (a_{n-1} + a_n) x^n$$
$$y' = a_1 + \sum_{n=1}^{\infty} (n+1) a_{n+1} x^n$$

SAME?

So $a_0 = a_1$
$$a_{n-1} + a_n = (n+1) a_{n+1}$$
$$a_{n+1} = \frac{a_{n-1} + a_n}{n+1}$$

Ex) Suppose $y(0)=1$, find $a_0, a_1, a_2, a_3, a_4$. (Using previous example.)

$a_1 = a_0 \qquad a_{n+1} = \dfrac{a_{n-1} + a_n}{n+1}$

$a_0 = 1 = a_1$

$n=1 = a_2 = \dfrac{a_0 + a_1}{1+1} = \dfrac{1+1}{1+1} = 1 = a_2$

$n=2 = a_3 = \dfrac{a_1 + a_2}{2+1} = \dfrac{1+1}{3} = \dfrac{2}{3} = a_3$

$n=3 = a_4 = \dfrac{a_2 + a_3}{3+1} = \dfrac{1+\frac{2}{3}}{4} = \dfrac{\frac{5}{3}}{4} = \dfrac{5}{12} = a_4$

So $y \approx 1 + 1x + 1x^2 + \frac{2}{3}x^3 + \frac{5}{12}x^4$
$\approx 1 + x + x^2 + \frac{2}{3}x^3 + \frac{5}{12}x^4$

HW-ish: Solve this ex a different way to get the explicit sol. then graph that sol and the one we just found.

$\frac{dy}{y} = (1+x)\,dx \Rightarrow \int \frac{dy}{y} = \int (1+x)\,dx \Rightarrow \ln(y) = x + \frac{1}{2}x^2 + C$

$y = e^{x + \frac{1}{2}x^2 + C}$ @ $y(0)=1$ $\quad 1 = e^{0+0+C} \Rightarrow C = 0$

$\boxed{y = e^{x + \frac{1}{2}x^2}}$