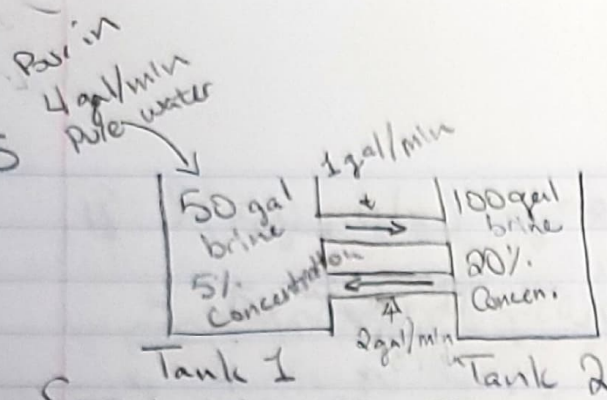
6/5



Time 0 { Tank 1 has 5% × 50 gallons = 2.5 grams of salt ⇒ 5% g/gal
Tank 2 has 20% × 100 gallons = 20 grams of salt ⇒ 20% g/gal

Want to find concentration as a function of time

$x_1(t)$ = amount of salt @ time t in tank 1

$x_2(t)$ = " " in tank 2

Concentration is salt / volume

$V_1(t)$ = volume of brine at time t in tank 1

$V_2(t)$ = " " tank 2

$$V_1(t) = 50 + 5t \quad , \quad V_2(t) = 100 - 5t$$

Concentrations: $c_1(t) = \frac{x_1(t)}{V_1(t)}$, $c_2(t) = \frac{x_2(t)}{V_2(t)}$

Change in x_1 ⇒ $\frac{dx_1}{dt}$ (grams/min)

$$= 0 - \frac{x_1(t)}{V_1(t)} \cdot 1 + \frac{x_2(t)}{V_2(t)} \cdot 2 \qquad \frac{g}{min} = \frac{g}{gal} \cdot \frac{gal}{min}$$

Salt in from water; $\frac{x_1(t)}{V_1(t)}$ Concentration tank 1; 1 gal away; 2 gal in; $\frac{x_2(t)}{V_2(t)}$ Concentration of tank 2

Change in x_2 ⇒ $\frac{dx_2}{dt}$

$$= \frac{x_1(t)}{V_1(t)} \cdot 1 - \frac{x_2(t)}{V_2(t)} \cdot 2 - \frac{x_2(t)}{V_2(t)} \cdot 4$$

$$= \frac{x_1(t)}{V_1(t)} - \frac{x_2(t)}{V_2(t)} \cdot 6$$

In from tank 1; out of tank 2

Rewrite dropping t for simplicity

$$x_1' = \frac{-x_1}{50+5t} + \frac{2x_2}{100-5t} \qquad x_2' = \frac{x_1}{50+5t} - \frac{6x_2}{100-5t}$$

and $x_1(0) = 2.5$

$x_2(0) = 20$

$$x_1' = \frac{-x_1}{50+5t} + \frac{2x_2}{100-5t} \qquad x_2' = \frac{x_1}{50+5t} - \frac{6x_2}{100-5t}$$

$$x_1(0) = 2.5, \; x_2(0) = 20$$

Now get x_2 in terms of x_1, x_1', t from the first eqn.

$$x_1' + \frac{x_1}{50+5t} = \frac{2x_2}{100-5t} \Rightarrow \frac{100-5t\left(x_1' + \frac{x_1}{50+5t}\right)}{2} = x_2$$

Then substitute x_2 into the second eq.
Which will give a second order ODE.

↓Easier version↓

Ex) Solve the system of ODE

$$\begin{cases} x_1' = 2x_1 + x_2 \\ x_2' = x_1 + 2x_2 \end{cases} \quad \text{with } x_1(0) = 1, \; x_2(0) = -1$$

$$\boxed{x_1' - 2x_1 = x_2} \Rightarrow x_1'(0) - 2(\overset{1}{x_1(0)}) = \overset{-1}{x_2(0)} \Rightarrow x_1'(0) = 1$$

$$x_1'' - 2x_1' = x_1 + 2x_1' - 4x_1$$

$$x_1'' - 4x_1' + 3x_1 = 0 \Rightarrow r^2 - 4r + 3 = 0$$

$$r = 1, 3$$

$$x_1 = c_1 e^t + c_2 e^{3t} \qquad @ \; x_1(0) = 1$$

$$x_1' = c_1 e^t + 3c_2 e^{3t} \qquad x_1'(0) = 1$$

$$1 = c_1 1 + c_2 (1)$$

$$-1 = c_1 1 + 3c_2 (1)$$

$$0 = -3c_1 \Rightarrow c_2 = 0$$

$$c_1 = 1$$

$$\boxed{x_1 = e^t} \Rightarrow x_1' - 2x_1 = x_2$$

$$x_1' = e^t \qquad e^t - 2e^t = x_2 \Rightarrow \boxed{-e^t = x_2}$$