From Last time:

$$\begin{cases} x_1' = 2x_1 + x_2 \\ x_2' = x_1 + 2x_2 \end{cases} \quad (I)$$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$(*)$ $X' = AX$, A is a 2×2 constant matrix.

If X_1 & X_2 are lin ind sets then all solutions to $(*)$ is of the form

$$X = c_1 X_1 + c_2 X_2$$

For the system (I),

$$X_1 = e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix}$$

$$X_2 = e^{3t}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix}$$

How to find X_1 and X_2 from the Matrix A?

Seek a solution to $(*)$ in the form

$$X = e^{rt} v$$

v is a constant 2×1 vector. (Not the 0 vector)

Plug into $(*)$.

$X' = re^{rt}v \qquad AX = A(e^{rt}v) = e^{rt}Av$

We need to choose $r \in \mathbb{R}$ and $v \in (2\times 1 \text{ vector})$ such that:

$$re^{rt}v = e^{rt}Av \Rightarrow Av \text{ will be a } 2\times 1 \text{ vector}$$

$$\tfrac{1}{e^{rt}}(\quad)\;(\quad)\tfrac{1}{e^{rt}}$$

$$rv = Av$$

$\Rightarrow r$ is an eigenvalue of A and v is an eigenvector associated with r.

So we get r and v from matrix A.

In this problem, $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

Now to find eigenvalues of A:

Notice $rv = Av$

$$Av - rv = 0 \Rightarrow Av - rI_2v = 0$$

$$(A - rI_2)v = 0$$

Still not 0

Solve the eq. $\det(A - rI_2) = 0$

$$A - rI_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - r\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$$

$$\det\begin{bmatrix} 2-r & 1 \\ 1 & 2-r \end{bmatrix} = 0$$

$$0 = (2-r)(2-r) - 1$$

$$0 = 4 - 4r + r^2 - 1 = r^2 - 4r + 3 = (r-3)(r-1)$$

$$r = 1, 3$$

$r_1 = 1$, $r_2 = 3$ and they do not have to share a v.

So v_1 goes with r_1 and v_2 goes with r_2.

r_1 & r_2 are eigenvalues, now find eigenvectors v_1.

$$(A - r_1 I_2)v_1 = 0 \quad \text{and} \quad (A - r_2 I_2)v_2 = 0$$

$$\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)v_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 - R_1} \begin{array}{cc} x & y \end{array} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

No pivot

$x + y = 0 \Rightarrow x = -y$

$y = y$

So $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ y \end{bmatrix} = y\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ | So $r_1 = 1$, $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$r_2 = 3 \Rightarrow (A - r_2 I_2) v_2 = 0$

$$\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc:c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \xrightarrow{R_2 + R_1} \left[\begin{array}{cc:c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad -x + y = 0 \quad x = y$$

$y = y$

$$v_2 = \begin{bmatrix} y \\ y \end{bmatrix} = y\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\boxed{r_2 = 3, \ v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

So $X_1 = e^{t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $X_2 = e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Therefore, all solutions to $(*)$ are of the form

$$X = c_1 X_1 + c_2 X_2$$
$$= c_1 e^{t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Given initial condition $X(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ how to find c_1 & c_2.

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = c_1\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Now we can solve a system of ODE by a matrix method.