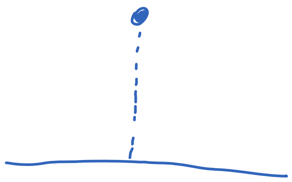
Lecture 5

Monday, April 10, 2023 10:19 AM

* Questions..

Some examples of autonomous ODEs

① Falling object under resistence :



$$ma = mg - kv^2$$

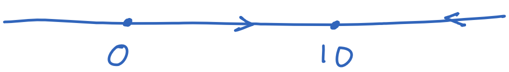
$$\rightsquigarrow mv' = mg - kv^2$$

$$\rightsquigarrow v' = g - \frac{k}{m} v^2$$

For simplicity, take $g = 10$, $\frac{k}{m} = \frac{1}{10}$:

$$v' = 10 - \frac{1}{10} v^2$$

Equilibrium state (or stationary point) is $v_* = 10$.



velocity keeps increasing but never gets above 10.

10 is the terminal velocity

② population model

Malthus's model (1798) $\underbrace{x' = kx}$

good for relatively short time

population grows exponentially

Verhurst's model (1838) : $x' = kx\left(1 - \frac{x}{M}\right)$

M is the sustainable population. Verhurst's model seems to be more realistic than Malthus's model.

How to solve?

$$\frac{x'}{x(1-\frac{x}{M})} = k \rightsquigarrow \frac{dx}{x(1-\frac{x}{M})} = k\,dt \rightsquigarrow \int \frac{dx}{x(1-\frac{x}{M})} = \int k\,dt$$

$$\rightsquigarrow \int \left(\frac{1}{x} + \frac{1/M}{1-\frac{x}{M}}\right) dx = kt + C$$

$$\rightsquigarrow \ln x - \ln\left(1-\frac{x}{M}\right) = kt + C$$

$$\rightsquigarrow \ln \frac{x}{1-\frac{x}{M}} = kt + C$$

$$\rightsquigarrow \frac{x}{1-\frac{x}{M}} = e^{kt+C} = \underbrace{e^{C}}_{\alpha} e^{kt} = \alpha e^{kt}$$

$$\rightsquigarrow x = \alpha e^{kt}\left(1-\frac{x}{M}\right) = \alpha e^{kt} - \frac{\alpha e^{kt}}{M} x$$

$$\rightsquigarrow x\left(1 + \frac{\alpha e^{kt}}{M}\right) = \alpha e^{kt}$$

$$\rightsquigarrow x = \frac{\alpha e^{kt}}{1 + \frac{\alpha e^{kt}}{M}}$$

There are 3 unknown parameters involving: α, k, M.

If we know the population at 3 different times, we will be able to determine α, k, M. After that, x is an explicit function of time.

For example, if we know the population of the U.S. in 1950, 1960, 1970, we will be able to use Verhurst's model to predict the population in 2030.