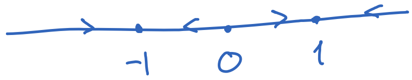
Lecture 6

Tuesday, April 11, 2023 3:49 PM

* Questions...

- Find an ODE whose phase portrait is as follows



$$x' = -x(x-1)(x+1)$$

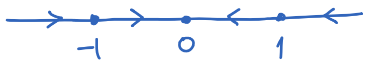
$$x' = -x^3(x-1)^3(x+1)^3$$

$$x' = \frac{-x(x-1)(x+1)}{1+x^2}$$

....

The phase portrait doesn't tell us the full story about the ODE because many different ODE can have the same phase portrait.

- Find an ODE whose phase portrait looks like this:



$$x' = -x(x-1)^2(x+1)^2$$

Separable equations

These are the ODEs of the form $x' = f(x)\,g(t)$.

Ex $\quad x' = f(x)$: function $g(t)$ is the constant 1. All autonomous ODE are also separable equations.

$$x' = xt$$

$$x' = \frac{x^2+1}{t^2+1} = (x^2+1)\frac{1}{t^2+1}$$

$$x' = e^{x+t} = e^x e^t$$

Non-examples:

$$x' = x+t, \quad x' = \sin(xt), \quad .$$

How to solve $x' = f(x)g(t)$?

Bring all x's to one side, t's to the other side

$$\frac{x'}{f(x)} = g(t)$$

Multiply both sides by dt:

$$\frac{dx}{f(x)} = g(t)\,dt$$

Integrate

$$\int \frac{dx}{f(x)} = \int g(t)\,dt$$

$\underline{\underline{\text{Ex}}}$

$$y' = \frac{2xy}{x^2+1}, \quad y(0) = 2$$

$$\rightsquigarrow \frac{y'}{y} = \frac{2x}{x^2+1} \rightsquigarrow \frac{dy}{y} = \frac{2x}{x^2+1}\,dx$$

$$\rightsquigarrow \int \frac{dy}{y} = \int \frac{2x}{x^2+1}\,dx = \ln(x^2+1) + C \quad (\text{substitution } u = x^2+1)$$

$$\rightsquigarrow \ln|y| = \ln(x^2+1) + C$$

Because $y(0) = 2 > 0$, we assume $y > 0$. Then $|y| = y$.

$$\ln y = \ln(x^2+1) + C$$

Exponentiate:

$$y = (x^2+1)e^C$$

Plug $x = 0$: $2 = (0^2+1)e^C \rightarrow e^C = 2$.

Therefore, $y = 2(x^2+1)$.

Ex $y' = \dfrac{2xy}{x^2+1}$, $y(0) = -2$

Similar to the above example $y = -2(x^2+1)$.

* Note. The difficulty of the separable equations doesn't lie only on the integration. Consider this example:

$$y' = \frac{x^2+1}{y^2+1}, \quad y(0) = 1$$

$$\rightsquigarrow (y^2+1)\, y' = x^2+1$$

$$\rightsquigarrow (y^2+1)\, dy = (x^2+1)\, dx$$

$$\rightsquigarrow \int (y^2+1)\, dy = \int (x^2+1)\, dx$$

$$\rightsquigarrow \frac{y^3}{3} + y = \frac{x^3}{3} + x + C$$

Plug $x = 0$:

$$\frac{1^3}{3} + 1 = \frac{0^3}{3} + 0 + C \rightsquigarrow C = \frac{4}{3}$$

$$\rightsquigarrow \frac{y^3}{3} + y = \frac{x^3}{3} + x + \frac{4}{3}$$

This is an implicit form of the solution. It is very difficult to go further from here. However, we can visualize $y = y(x)$ on Mathematica.