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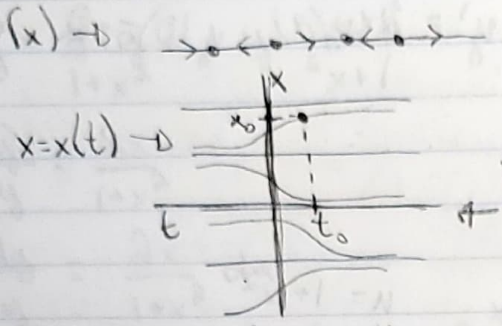
$x' = f(x) \rightarrow$



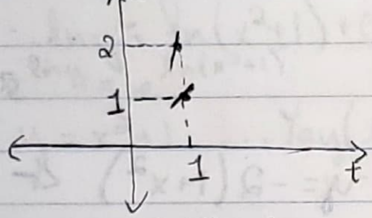
Mathematica can help with this sketch.

at position (x_0, t_0), there is a curve $x = x(t)$ passing through it.

The slope of this curve at this position is $x'(t_0)$

And $x'(t_0) = f(x(t_0)) = f(x_0)$

Ex) $f(x) = x^2$



$f(x_0) = f(1) = 1$

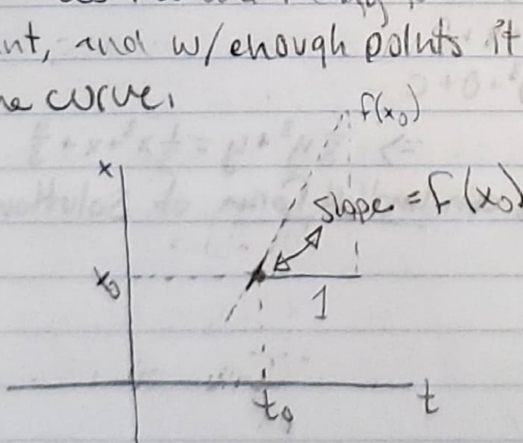
$= f(2) = 4$

Direction Field $\rightarrow$ (If lots of segments)

If you do lots of these little lines you get a rough picture of the curve, which you could connected to get a curve.

That what Mathematica does!

It takes however many points in an area you want, and w/enough points it can connected the curve.



@ position, (t_0, x_0), draw vector $(1, f(x_0))$

Ex) (on Mathematica)

[ ]: f[x_]:= Sin[x]

VectorPlot [{1, f[x]}, {t, -2, 2}, {x, -4, 4},

VectorPoint -> 50]

Streamplot[" "]

Linear First Order ODE

$x' + p(t)x = q(t)$ where $p(t)$ & $q(t)$ are given.

$p(t) = p$ & $q(t) = q$

Ex) $x' + tx = t^2$

$tx' + x = e^t$ ← (Not in standard form)

↳ $x' + \frac{1}{t} = \frac{e^t}{t}$

Multiply both sides of = by a quantity $\underbrace{u = u(t)}_{\text{TBD}}$

$(x' + px = q)u$

$\underbrace{ux' + upx}_{(ux)'} = qu$

Want to design a u so that $ux' + upx = (ux)'$

$= ux' + u'x$

So $up = u'$ ← Want this.

That mean $u' = ut$ (which we integrate like its seperable)

$\frac{u'}{u} = p \cdot dt \quad \frac{du}{u} = p\,dt\,dt$

$\int \frac{du}{u} = \int p\,dt \rightsquigarrow \ln u = \int p(t)\,dt$

$u = e^{\int p(t)dt}$

↖ This is the u we need!

↖ Integrating Factor

$(ux)'$

$ux = \int qu\,dt$

$x = \frac{1}{u}\int qu\,dt$

Ex) $x' + 3x = t$

Integrating Factor: $\int p(t)\,dt = \int 3\,dt = 3t + C$

$\hookrightarrow u = e^{3t}$ ← No C because we only need 1 integrating factor

$x = \frac{1}{u} = \frac{1}{e^{3t}} \int t e^{3t}\,dt$ (No cancelling here!!!)

$\underbrace{e^{-3t} \int t e^{3t}\,dt}_{HW!}$

Summary: $y' + p(t)y = g(t)$

Integrating factor is $u = e^{\int p(t)\,dt}$

Don't have to include C for the u, do need it in the solution

Solution: $y = \frac{1}{u} \int u g(t)\,dt$

Ex) $2xy' - 3y = 9x^3$

$y' - \frac{3y}{2x} = \frac{9x^3}{2x}$

$y' - \frac{3}{2x} y = \frac{9}{2} x^2$

Integrating factor $u = e^{\int -\frac{3}{2x}\,dx}$

$\frac{-3}{2} \int \frac{1}{x}\,dx = \frac{-3}{2} \ln(x) = u$

$e^{\frac{-3}{2}\ln(x)} = x^{\frac{-3}{2}} = u$

$= \frac{1}{x^{3/2}}$

$y = x^{3/2} \int \frac{1}{x^{3/2}} \frac{9}{2} x^2\,dx$

$= \frac{9}{2} x^{3/2} \int \frac{x^2}{x^{3/2}}\,dx$

$= \frac{9}{2} x^{3/2} \int x^{1/2}\,dx$

$y = \frac{9}{2} x^{3/2} \left(\frac{2}{3} x^{3/2} + C \right)$

$y = 3x^3 + \frac{9}{2} x^{3/2} C$

Checking: $y' = 9x^2 + \frac{27C}{4} x^{1/2}$

$\left(9x^2 + \frac{27C}{4} x^{1/2}\right) - \frac{3\left(3x^3 + \frac{9C}{2} x^{3/2}\right)}{2x} = \frac{9}{2} x^2$

$-\frac{\left(9x^3 + \frac{27C}{2} x^{3/2}\right)}{2x} = -\frac{\left(9x^2 + \frac{27C}{2} x^{1/2}\right)}{2}$

$9x^2 + \frac{27C}{4} x^{1/2} - \left(\frac{9}{2} x^2 - \frac{27C}{4} x^{1/2}\right) = \frac{9}{2} x^2$

$-\frac{9}{2} x^2 \qquad -\frac{9}{2} x^2$

$9x^2 + \frac{27C}{4} x^{1/2} - 9x^2 - \frac{27C}{4} x^{1/2} = 0$ ✓