Ex) $x' + 3x = t$

Integrating Factor: $\int p(t)\,dt = \int 3\,dt = 3t + C$

$\hookrightarrow u = e^{3t}$ ← No C because we only need 1 integrating factor

$x = \frac{1}{u} = \frac{1}{e^{3t}} \int t e^{3t}\,dt$ (No cancelling here!!!)

$e^{-3t} \underbrace{\int t e^{3t}\,dt}_{HW!}$

Summary: $y' + p(t)y = g(t)$

Integrating Factor is $u = e^{\int p(t)\,dt}$

Don't have to include C for the u, do need it in the solution

Solution: $y = \frac{1}{u} \int u g(t)\,dt$

Ex) $2xy' - 3y = 9x^3$

$y' - \frac{3y}{2x} = \frac{9x^3}{2x}$

$y' - \frac{3}{2x} y = \frac{9}{2} x^2$

Integrating factor $u = e^{\int -\frac{3}{2x}\,dx}$

$\frac{-3}{2} \int \frac{1}{x}\,dx = \frac{-3}{2} \ln(x) = u$

$e^{\frac{-3}{2}\ln(x)} = x^{\frac{-3}{2}} = u$

$= \frac{1}{x^{3/2}}$

$y = x^{3/2} \int \frac{1}{x^{3/2}} \frac{9}{2} x^2\,dx$

$= \frac{9}{2} x^{3/2} \int \frac{x^2}{x^{3/2}}\,dx$

$= \frac{9}{2} x^{3/2} \int x^{1/2}\,dx$

$y = \frac{9}{2} x^{3/2} \left(\frac{2}{3} x^{3/2} + C \right)$

$y = 3x^3 + \frac{9}{2} x^{3/2} C$

Checking: $y' = 9x^2 + \frac{27C}{4} x^{1/2}$

$\left(9x^2 + \frac{27C}{4} x^{1/2}\right) - \frac{3\left(3x^3 + \frac{9C}{2} x^{3/2}\right)}{2x} = \frac{9}{2} x^2$

$\frac{-\left(9x^3 + \frac{27C}{2} x^{3/2}\right)}{2x} = \frac{-\left(9x^2 + \frac{27C}{2} x^{1/2}\right)}{2}$

$9x^2 + \frac{27C}{4} x^{1/2} - \left(\frac{9}{2} x^2 - \frac{27C}{4} x^{1/2} \right) = \frac{9}{2} x^2$

$-\frac{9}{2} x^2 \qquad -\frac{9}{2} x^2$

$9x^2 + \frac{27C}{4} x^{1/2} - 9x^2 - \frac{27C}{4} x^{1/2} = 0$ ✓

Ex) $x^2 y' + 2xy = \sin x$

$$y' + \frac{2xy}{x^2} = \frac{\sin x}{x^2}$$

$$y' + \frac{2}{x} y = \frac{\sin x}{x^2}$$

$$u = e^{\int p\,dt} \Rightarrow u = e^{\int \frac{2}{x} dx}$$

$$= e^{2\ln x} \Rightarrow x^2 = u$$

$$y = \frac{1}{x^2} \int x^2 \frac{\sin x}{x^2} dx$$

$$y = \frac{1}{x^2} \int \sin x\, dx$$

$$y = \frac{-\cos x + C}{x^2} \Rightarrow (-\cos x + c)(x^{-2})$$

$$(\sin x)(x^{-2}) + (-\cos x + c)(-2x^{-3})$$

$$\frac{\sin x}{x^2} + \frac{(-\cos x + c)(-2)}{x^3}$$

$$y' = \frac{x^3 \sin x + 2x^2 \cos x - 2cx^2}{x^5}$$

$$y' = \frac{x \sin x + 2\cos x - 2c}{x^3}$$

$-2\cos x + 2c$

$$\frac{x \sin x + 2\cos x - 2c}{x^3} + \frac{2}{x} \cdot \frac{-\cos x + C}{x^2} = \frac{\sin x}{x^2}$$

x^3

$$\frac{x \sin x}{x^3} = \frac{\sin x}{x^2}$$

$$\checkmark \quad \frac{\sin x}{x^2} = \frac{\sin x}{x^2}$$

Application: Newtons Law of Cooling:

ball @ temp = T, enviroment temp = t_0

$T > t_0 \Rightarrow T(t_0)\downarrow$ → The greater the difference, the faster the change

$T < t_0 \Rightarrow T(t_0)\uparrow$ →

is proportional

$$T'(t_0) \sim (T_0 - T(t))$$

$$T'(t) = k(T_0 - T(t))$$