Lecture 12

Friday, May 31, 2024 2:40 PM

Addition of angles:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

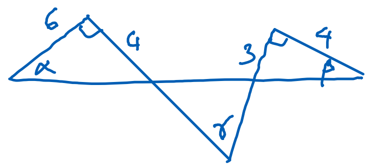
Moral: if we know sin, cos of x and y, we'll know everything about $x+y$.

Ex find $\sin 75^\circ$, $\cos 105^\circ$

$$\sin(75^\circ) = \sin(30^\circ + 45^\circ) = \sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ = \frac{1}{2}\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} = \frac{\sqrt{2}+\sqrt{6}}{4}.$$

$$\cos(105^\circ) = \cos(45^\circ + 60^\circ) = \dots$$

Ex



$\gamma = ?$

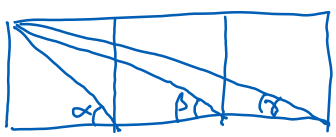
Note that $\gamma = \alpha + \beta$.

$$\tan\gamma = \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}$$

$$= \frac{\frac{4}{6} + \frac{3}{4}}{1 - \frac{4}{6}\frac{3}{4}} = \frac{17}{6}$$

Thus, $\gamma = \tan^{-1}\left(\frac{17}{6}\right) \approx \dots$

Ex



show that $\beta + \gamma = \alpha$

Notice that $\alpha = 45^\circ$, $\tan\beta = \frac{1}{2}$, $\tan\gamma = \frac{1}{3}$.

$$\tan(\beta+\gamma) = \frac{\tan\beta + \tan\gamma}{1 - \tan\beta\tan\gamma} = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\frac{1}{3}} = 1$$

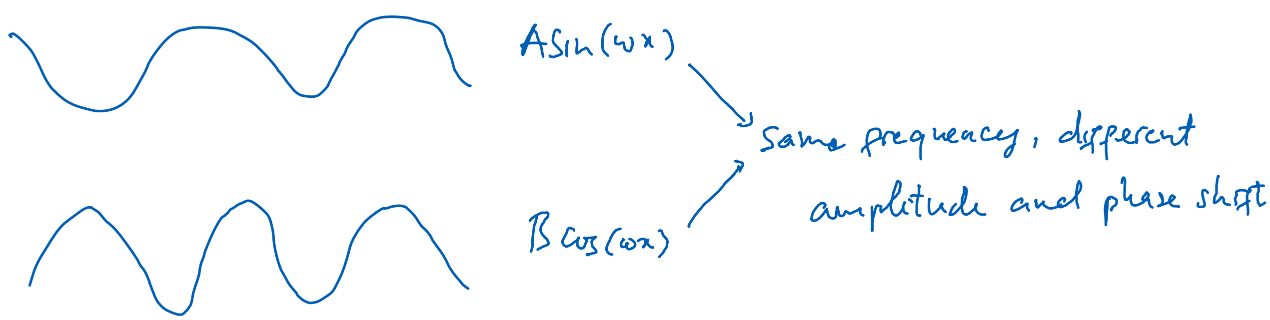
Thus, $\beta + \gamma = \tan^{-1}(1) = 45^\circ = \alpha$.

Ex Show that $\sin(2x) = 2\sin x\cos x$

$$\cos(2x) = 2\cos^2 x - 1$$

Hint: $2x = x + x$

Add two waves with the same frequency:



$$A\sin(\omega x) + B\cos(\omega x) = \sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\sin(\omega x) + \frac{B}{\sqrt{A^2+B^2}}\cos(\omega x)\right)$$

Find ϕ such that $\cos\phi = \frac{A}{\sqrt{A^2+B^2}}$, $\sin\phi = \frac{B}{\sqrt{A^2+B^2}}$

Then

$$A\sin(\omega x) + B\cos(\omega x) = \sqrt{A^2+B^2}\left(\cos\phi\sin(\omega x) + \sin\phi\cos(\omega x)\right)$$

$$= \underbrace{\sqrt{A^2+B^2}}_{\text{new amplitude}}\sin(\omega x + \phi)$$

new phase shift