Lab 2

This lab is built on the previous lab (Lab 1). *The next lab will be built on this lab, so please make sure that you go through all the instruction carefully and do all the assignments.* In this lab, you will learn how to:

- graph an equation,

- graph a list of points,

- graph parametric curves,

- compute length and curvature of a curve.

1 To turn in

Do Problems 1-27 in a single Mathematica Notebook file, or ipynb file if you use JupyterLab. Write *your name* and *lab number* at the beginning of your report. Clearly label each problem to separate them from each other. Make sure to comment on each problem. If your code doesn't run correctly, explain what you are trying to do. Submit on Canvas both the *pdf file* and the *source file* (nb or ipynb). I will re-run all commands when I grade. Here is the breakdown of points:

Problems	Points
3, 7, 10-12, 15, 16, 23	1
1, 2, 4, 9, 13, 17, 19, 20, 25	2
5, 6, 8, 18, 21, 22, 24, 26, 27	3
14	4
Readability of your report	3
Total: 27	Total: 60

2 Graph an equation

Suppose two quantities, x and y, are related to each other by an equation. Examples of such an equation include:

$$y = \sin(x) \tag{1}$$
$$x^2 + 2y = 5 \tag{2}$$
$$ye^{x+y} = 1 \tag{3}$$

Each pair (x, y) satisfying the equation corresponds to a point on the plane. To graph the equation is to put all such points on the plane. If there is a single y on the left hand side of the equation and no occurrences of y on the right hand side (such as in Equation 1), then we can use the `Plot` command to show a graph of y. In this case, graphing the equation is simply graphing a function.

1. For example, to graph Equation 1, try the following:

   ```
   Plot[Sin[x],{x,0,4*Pi}]
   ```

   Can you graph the equation $y = \ln(e^x + 1)$, where x ranges from -5 to 5 ?

2. In Equation 2, although both x and y are on the same side of the equation, you can easily solve for y as a function of x and then graph it with `Plot`. Plot y as a function of x, where x ranges from -3 to 3.

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3. In Equation 3, it is very difficult to solve for y as a function of x. In this case, we will graph the equation as it is by the command `ContourPlot`. Try the following:

```
ContourPlot[y*E^(x+y)==1,{x,-2,2},{y,0,10}]
```

   Notice the double equal sign. Mathematica will try to solve numerically for the pairs (x, y), with $-2 \le x \le 2$ and $0 \le y \le 10$, that satisfy Equation 3 and then put them on the plane.

4. Graph the equation $\ln(x^2 y) = \cos(xy) + 1$. Choose your own range of x and y to get a good-looking graph.

5. The following equation represents an the ellipse

$$\frac{x^2}{4} + \frac{y^2}{16} = 1$$

   Draw this ellipse in two following ways. First, use `ContourPlot` with the range $-4 \le x, y \le 4$. Second, solve for y in terms of x and then use the command `Plot`. Note that you will have two values of y as a function of x (corresponding to plus and minus signs). Make sure to plot both of them on the same plot (see Exercise 24 of Lab 1).

6. Equations of the form

$$y^2 = x^3 + ax + b$$

   represent *elliptic curves*. Elliptic curves are an important area in Number Theory. In fact, the British mathematician Andrew Wiles used them to prove the notorious Fermat's Last Theorem. For $a = -2$ and for each value of $b = 2, 1, 0$, plot the elliptic curve where $-3 \le x, y \le 3$. For $a = 0$ and for each value of $b = 1, 0, -1$, plot the elliptic curve where $-3 \le x, y \le 3$.

7. The command `ContourPlot3D` plots an equation involving 3 variables, say x, y, and z. Mathematica will put all the points (x, y, z) satisfying the equation in the space. For example, to plot the ellipsoid

$$\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$$

   try the following:

```
ContourPlot3D[x^2/4+y^2/9+z^2==1,{x,-3,3},{y,-3,3},{z,-3,3}]
```

   Add the option `AxesLabel -> {"x", "y", "z"}` to label the axes.

8. Plot the following surfaces using `ContourPlot3D`. Choose your own ranges for x, y, z to get a good-looking picture.

   (a) $x^2 - y + z^2 = 1$
   (b) $z = x^2 - y^2$
   (c) $xy + yz + zx = 1$

   For Part (a) and (b), also plot using `Plot3D`. If you forget this command, review Exercise 25 and 26 of Lab 1.

3   Graph a list of points

A point *on the plane* may be viewed as a pair of numbers: the x-coordinate and the y-coordinate. You are familiar with the notation (x, y). In Mathematica, you use the *list notation* {} to represent a point. For example, {2,5} represents the point with coordinates $x = 2, y = 5$, and {2,5,3} represents the point *in the space* with coordinates $x = 2, y = 5, z = 3$.

The notation {{1,2},{2,-1},{3,0},{2,2}} represents a list of four points on the plane with coordinates $(1, 2), (2, -1), (3, 0), (2, 2)$. The notation {{1,2,3},{2,2,-1}} represents a list of two points in the space with coordinates $(1, 2, 3), (2, 2, -1)$.

9. To graph the list a points, we use the command `ListPlot`. For example, to graph the list {{1,2},{2,-1},{3,0},{2,2}}, try the following command:

    ```
    ListPlot[{{1,2},{2,-1},{3,0},{2,2}}]
    ```

    To change the color of the points, you can add a style option as follows:

    ```
    ListPlot[{{1,2},{2,-1},{3,0},{2,2}},PlotStyle->Red]
    ```

    You can also customize the size of the points. For example, try replacing the `PlotStyle` option above with `PlotStyle->{Red,PointSize[0.02]}`.

10. You can graph multiple lists of points on the same plot. For example, the two lists

    ```
    list1={{1,2},{2,-1},{3,0},{2,2}}
    list2={{3,2},{2,4},{2,1}}
    ```

    can be put on the same plot as follows:

    ```
    ListPlot[{list1,list2},PlotMarkers->Automatic,
          AspectRatio->Automatic,AxesOrigin->{0,0}]
    ```

11. You can generate a systematic list of points using the command `Table`. For example, the command

    ```
    Table[{t,Sin[t]},{t,0,2Pi,0.5}]
    ```

    generates a list of points of coordinates $(t, \sin t)$ where t runs from 0, increasing by a 0.5 each step, until it exceeds $2\pi \approx 6.28$. You can arrange this list into a table by the command

    ```
    Table[{t,Sin[t]},{t,0,2Pi,0.5}] // TableForm
    ```

    or equivalently, `TableForm[Table[{t,Sin[t]},{t,0,2Pi,0.5}]]`. To graph this list of points, do the following:

    ```
    list=Table[{t,Sin[t]},{t,0,2Pi,0.5}];
    ListPlot[list]
    ```

12. You can fit these points on the sine curve. To visualize this, you first graph the sine curve using `Plot`, then graph the list of points using `ListPlot`, and then show them together using `Show`.

```
p1=Plot[Sin[t],{t,0,2Pi}];
p2=ListPlot[list,PlotStyle->Red];
Show[p1,p2]
```

13. You can link consecutive points in the list by straight lines by adding the option `Joined->True` to the `ListPlot` command. For example, try the following:

    ```
    ListPlot[list,PlotStyle->Red,Joined->True]
    ```

    Can you draw this zigzag together with the sine curve on the same plot?

14. Based on Problems 11-13, do the following tasks:

    (a) Draw a list of points $(x, \ln x \sin x)$ where x runs from 1 to 10 with step size 1.
    (b) Fit these points on the curve $y = \ln x \sin x$.
    (c) Connect these points by straight lines to get a zigzag.
    (d) Show this zigzag together with the curve $y = \ln x \sin x$.

4   Parametric curves

A curve can be given by a parametric equation. A curve *on a plane* (or 2D curve) is of the form $r(t) = (x(t), y(t))$. A curve *in the space* (or 3D curve) is of the form $r(t) = (x(t), y(t), z(t))$. To draw them, you can use the commands `ParametricPlot` and `ParametricPlot3D`, respectively.

15. For example, to plot the curve $r(t) = (t - 2, t^2)$, try the following:

    ```
    ParametricPlot[{t-2, t^2}, {t, -2, 4}]
    ```

    To plot the curve $r(t) = (t \cos t, t \sin t, t)$, try the following:

    ```
    ParametricPlot3D[{t*Cos[t], t*Sin[t], t}, {t, 0, 30}]
    ```

    If you use Jupyter Notebook, add the prefix Interact@ to the command so that you can rotate the curve.

16. The parametric equations also show us how a curve is drawn. For example, try the following:

    ```
    Manipulate[ParametricPlot3D[{t*Cos[t], t*Sin[t], t}, {t, 0, s},
      PlotRange -> {{-30, 30}, {-30, 30}, {0, 30}}], {s, 0.1, 30}]
    ```

    If you use Jupyter Notebook, add the prefix Interact@ to the command to be able to interact with the curve.

17. Use the syntax in Problems 15 and 16 to plot the curve $r(t) = (\cos t, \sin t, \cos(2t))$ for $0 \le t \le 2\pi$.

18. Use the syntax in Problems 15 and 16 to plot the curve $r(t) = (-2 \cos t + \frac{4}{3} \cos(3t), \ 2 \sin t + \frac{4}{3} \sin(3t))$ for $0 \le t \le 2\pi$. For what (approximate) values of $t \in [0, 2\pi]$ does the curve intersect itself? (This curve is on the logo of the Proximus company.)

19. Find a parametrization of the 2D curve given by the equation $x^2 + xy + 2y = 1$ and plot it. *Hint: set $x = t$ and then find y in terms of t.*

5 Length and curvature

Recall the formula for length and curvature of a curve parametrized by $r(t)$, $a \leq t \leq b$:

$$L = \int_a^b |r'(t)|dt, \quad \kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

20. Let us consider an example where a curve is parametrized by $x = e^t$, $y = \sin(\pi t)$, $z = (t+1)^2$, $-2 \leq t \leq 1$. We want to find the length of the curve. First, we define the vector function $r(t) = (x, y, z) = (e^t, \sin(\pi t), (t+1)^2)$.

    ```
    r[t_]:={E^t,Sin[Pi t],(t+1)^2}
    ```

    In Mathematica, the differential of this vector function is $r'[t]$. Note that the norm (magnitude) of a vector u can be computed as $|u| = \sqrt{u \cdot u}$. Thus, the norm of $r'(t)$ can be computed in Mathematica as `Sqrt[r'[t].r'[t]]`. Now we are ready to put together a command to find the length of the curve:

    ```
    NIntegrate[Sqrt[r'[t].r'[t]],{t,-2,1}]
    ```

21. Plot the curve parametrized by $x = \cos^3(t)$, $y = \sin^3(t)$ and compute its length. You will need to determine a correct range of t so that the curve is closed and non-overlapping. This curve is called an astroid.

22. Plot the curve parametrized by $x = \cos(t) - \sin^2(t)$, $y = \cos(t)\sin(t)$ and compute its length. You will need to determine a correct range of t so that the curve is closed and non-overlapping. This curve is called a fish curve.

23. The curve in Problem 20 passes through the point $(1, 0, 1)$ because $r(0) = (1, 0, 1)$. To compute the curvature of the curve at this point, we use the formula

$$\kappa(0) = \frac{|r'(0) \times r''(0)|}{|r'(0)|^3}$$

    Besides using the square root of dot product, you can also compute the norm of a vector u by the command `Norm[u]`. We can evaluate the above as follows:

    ```
    Norm[Cross[r'[0],r''[0]]]/Norm[r'[0]]^3
    ```

24. The fish curve in Problem 22 is a 2D curve. But one can also view it as a 3D curve (with the z-coordinates equal to 0). Can you redefine the fish curve (by writing `r[t_]:=...`) so that it is a 3D curve?

25. Find the curvature of the fish curve as a function of t. Name this function as `k[t]`.

26. Find the points on the fish curve where the curvature are maximum or minimum with:

    ```
    FindMaximum[k[t],t]
    FindMinimum[k[t],t]
    ```

    You will see that Mathematica returns the maximum and minimum values of the fucntion `k[t]` together with the values of t at which the maximum and minimum are attained. To find the position on the curve, you will need to apply the function r to those values of t.

27. Put the position(s) with maximum curvature and the position(s) with minimum curvature on the fish curve. The position(s) of maximum curvature is in red and the position with minimum curvature is in orange.