Lab 4

This lab assignment is based on Lab 3. You will learn how to:

- solve an equation or system of equations,

- find min/max of a function,

- sketch polygons, 2D regions, and 3D solid,

- evaluate double and triple integrals.

1 To turn in

Do Problems 1-34 in a single Mathematica Notebook file, or ipynb file if you use JupyterLab. Write *your name* and *lab number* at the beginning of your report. Clearly label each problem to separate them from each other. Make sure to comment on each problem. If your code doesn't run correctly, explain what you are trying to do. Submit on Canvas both the *pdf file* and the *source file* (nb or ipynb). I will re-run all commands when I grade. Here is the breakdown of points:

Problems	Points
1, 11-13, 21-26, 29, 32	1
2, 5, 9, 10, 14-17, 27, 28	2
3, 4, 6-8, 18-20, 30, 31, 33, 34	3
Readability of your report	3
Total: 34	Total: 71

2 Solve a system of equations

1. The command **Solve** is used to solve an equation or a system of equations. For example, to find numerically solutions to the equation $x^2 - 4x - 6 = 0$, try the following:

   ```
   Solve[ x^2-4x-6==0, x]
   ```

   Note that the double equal sign to represent an equation.

2. To solve a system of equations, we list all the equations of the system inside the command `Solve`, separated by `&&`. Try the following:

   ```
   Solve[x+y==5 && x^2-y==3, {x,y}]
   ```

   To get numerical values instead of the exact solution, replace `Solve` by `NSolve`.

3. Find numerically all real roots of the polynomial $f(x) = x^5 - 10x - 5$. Graph the function f to verify your answer.

4. Show the curve $x + y^2 = 1$ and the curve $x^2 = y^3$ on the same plot. Then find all the intersection points between the two curves.

3 Optimization problem

5. Let $f(x,y) = y^3 + 3x^2y - 6x^2 - 6y^2 + 2$. Plot the function in the region $-5 \leq x \leq 5$, $-5 \leq y \leq 5$.

6. Draw a contour map with 200 level sets. How many critical points do you see? How many local min/ local max/ saddle points are there? Explain your answer.

7. Use the command **Solve** as explained above to find all the critical points of f. Based on the contour map, which one(s) are local min/ local max/ saddle points?

8. With the help of the contour map, determine approximately the maximum and minimum values of f in the region $-5 \leq x \leq 5$, $-5 \leq y \leq 5$ and the points (x,y) at which the function attains minimum and maximum.

4 Sketch regions and solids

9. To create a polygon (triangle, quadrilateral, pentagon, hexagon,...), we use the syntax

```
Polygon[{p1, p2, p3,...}]
```

where `p1,p2,p3,...` are the vertices of the polygon in that order. The command `Region` can be used to plot the region. Try the following:

```
R=Polygon[{{1,1},{2,1},{3,2},{-1,3}}]
Region[R]
```

For better visualization, add the options `Axes->True` and `AxesOrigin->{0,0}` to the command `Region`.

10. Plot the triangle with vertices at $(-1,1)$, $(2,0)$, $(1,2)$.

11. You can also plot a polygon in 3D. Try the following:

```
S=Polygon[{{1,0,3},{0,2,3},{1,2,0}}]
Region[S, Axes->True,
          AxesOrigin->{0,0,0},
          AxesStyle->Thick,
          AxesLabel->{x,y,z}]
```

12. Regions and solids are usually described in terms of inequalities. For example, the ellipse centered at the origin, meeting the x-axis at $x = \pm 4$ and the y-axis at $y = \pm 3$ is described as

$$\frac{x^2}{16} + \frac{y^2}{9} \leq 1.$$

You can define this ellipse using `ImplicitRegion` and plot it using `Region` as follows:

```
R = ImplicitRegion[x^2/16+y^2/9<=1, {x,y}]
Region[R, Axes->True, AspectRatio->Automatic, AxesLabel->{x,y}]
```

13. The part the the ellipse in the first quadrant is described by the inequalities

$$\frac{x^2}{16} + \frac{y^2}{9} \le 1, \ x \ge 0, \ y \ge 0.$$

These inequalities are separated by `&&` as follows.

```
R = ImplicitRegion[x^2/16+y^2/9<=1 && x>=0 && y>=0, {x,y}]
Region[R, Axes->True, AspectRatio->Automatic, AxesLabel->{x,y}]
```

14. Plot the upper half of the unit disk centered at the origin.

15. Plot the region above the parabola $y = x^2$, below the line $y = 4$, and on the right of the y-axis.

16. Plotting a solid is a bit trickier, not because it can't be done the same way as for regions, but because we want good visual effects. Let R be the solid between two surfaces $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$. This solid can be defined as

```
R = ImplicitRegion[x^2+y^2<=z<=8-x^2-y^2, {x,y,z}]
```

You then use the command `RegionPlot3D` to plot this solid:

```
RegionPlot3D[R, Axes->True,
                Mesh -> False,
                AxesOrigin->{0,0,0},
                AxesLabel->{x,y,z},
                PlotStyle->Opacity[0.5],
                AxesStyle->Thick,
                Boxed->False]
```

Adjust some parameters in the command (for example, `Boxed->True`, `Opacity[0.1]`,...) to see what happens. If you use Jupyter Notebook, add the prefix `Interact@` in front of `RegionPlot3D` to be able to rotate the solid.

17. Another way to plot the solid is to draw its boundaries on the same plot, which are the paraboloids $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$. Try the following:

```
Plot3D[{x^2+y^2,8-x^2-y^2}, {x, -2, 2}, {y,-2, 2}]
```

To enhance the visual effect, add the following options (separated by commas):

```
PlotStyle -> {{Blue, Opacity[0.5]}, {Yellow, Opacity[0.4]}}
AxesLabel -> {Style["x",Bold,16],Style["y",Bold,16],Style["z",Bold,16]}
Mesh -> None
AxesOrigin->{0,0,0}
PlotRange->{{-2,2},{-2,2},{0,9}}
Boxed->False
BoxRatios->Automatic
```

You can see that the two surfaces have extra pieces sticking out of the solid. You can cut off these extra pieces by adding the following option to the previous command:

```
RegionFunction->Function[{x,y,z},x^2+y^2<=4]
```

18. Plot the solid inside the ellipsoid $x^2 + 4y^2 + z^2 = 9$ and in the first octant (i.e. $x, y, z \geq 0$).

19. Plot the solid under the plane $z = 1 + x + y$, above the region in the xy-plane bounded by the curves $y = \sqrt{x}$, $y = 0$, and $x = 1$.

20. Plot the solid enclosed by the cylinder $x^2 + z^2 = 4$ and the planes $y = -1$ and $y + z = 4$.

5 Double and triple integrals

The command **Integrate** is used to compute the exact integrals (definite or indefinite) in Mathematica. The command **NIntegrate** is used to compute approximate numerical value of a definite integral. For example,

21. $\int \sin x dx$

```
Integrate[Sin[x], x]
```

22. $\int_0^\pi \sin x dx$

```
Integrate[Sin[x], {x, 0, Pi}]
```

23. $\int_0^2 \int_0^1 (x + 2y) dx dy$

```
Integrate[x + 2 y, {y, 0, 2}, {x, 0, 1}]
```

24. $\int_0^2 \int_{y-1}^{2y} (x + 2y) dx dy$

```
Integrate[x + 2 y, {y, 0, 2}, {x, y - 1, 2 y}]
```

25. Numerical approximation of $\int_0^2 \int_0^1 e^{-x^2 y^2} dx dy$

```
NIntegrate[Exp[-x^2 y^2], {x, 0, 1}, {y, 0, 2}]
```

26. $\iint_R (x + y)) dA$ where R is the quadrilateral with vertices at $(1, 1), (2, 1), (3, 2), (-1, 3)$

```
R=Polygon[{{1,1},{2,1},{3,2},{-1,3}}]
Integrate[x+y, {x,y} \[Element] R]
```

As you type \[Element], it will be turned into the symbol $\in$, which is read "is a member of" or "belongs to". You can also type this symbol by typing **ESC el ESC** on your keyboard.

27. Find the precise value and numerical value of the integral

$$\int_{-3}^3 \int_0^{\pi/2} (y + y^2 \cos x) dx dy$$

28. Find the precise value and numerical value of the integral

$$\iint_D \frac{y}{x^5 + 1} dA$$

where $D = \{(x, y) : \ 0 \leq x \leq 1, 0 \leq y \leq x^2\}$.

29. To find
$$\int_0^1 \int_0^x \int_x^{x+y} (x+z)\,dz\,dy\,dx$$

we use the command

```
Integrate[x+z, {x,0,1}, {y,0,x}, {z,x,x+y}]
```

Notice the order of integration in the command.

30. Find the precise value and numerical value of the integral
$$\int_0^2 \int_0^{z^2} \int_0^{y-z} (2x-y)\,dx\,dy\,dz$$

Sketch the solid over which the integral is taken.

31. Find the precise value and numerical value of the integral
$$\int_0^1 \int_0^1 \int_0^{\sqrt{1-z^2}} \frac{z}{y+1}\,dx\,dz\,dy$$

Sketch the solid over which the integral is taken.

32. You can define the solid before integrating. For example, to find $\iiint_R y\,dV$ where R is the solid is given by
$$R = \{(x,y,z): \ 0 \le y \le 1, \ y \le x \le 1, \ 0 \le z \le xy\}$$

you can do the following:

```
R=ImplicitRegion[0<=y<=1 && y<=x<=1 && 0<=z<=x*y,{x,y,z}]
Integrate[y, {x,y,z} \[Element] R]
```

33. Use the method in Problem 32 to find the precise value and numerical value of the integral
$$\iiint_E \frac{z}{x^2+z^2}\,dV$$

where $E = \{(x,y,z): \ 1 \le y \le 4, \ y \le z \le 4, \ 0 \le x \le z\}$. Sketch the solid E.

34. Compute $\iint_R \frac{1}{x^2+y^2+1}\,dA$ where R is the triangle with vertices at $(0,1),(4,5),(3,2)$.