

## Lab 5

This lab assignment is based on Lab 4. You will learn how to:

- find the Jacobian matrix and determinant of a change of variables,
- sketch the new region after the change of variables,
- visualize 2D and 3D vector fields,
- compute the curl and divergence of a vector field,
- compute line integrals and surface integrals.

### 1 To turn in

Do Problems 1-24 in a single Mathematica Notebook file, or ipynb file if you use JupyterLab. Write *your name* and *lab number* at the beginning of your report. Clearly label each problem to separate them from each other. Make sure to comment on each problem. If your code doesn't run correctly, explain what you are trying to do. Submit on Canvas both the *pdf file* and the *source file* (nb or ipynb). I will re-run all commands when I grade. Here is the breakdown of points:

Problems	Points
1, 6, 14, 21	1
4, 5, 9, 10, 11, 13, 16, 17, 18, 22, 23	2
3, 7, 8, 12, 15, 19, 20, 24	3
Readability of your report	3
Total: 24	Total: 55

### 2 Find the Jacobian of a change of variables

Consider the change of variables  $(x, y) \rightarrow (u, v)$ , where  $(x, y)$  are the old variables and  $(u, v)$  are the new variables. Equivalently, you can call it the *transformation*  $(x, y) \rightarrow (u, v)$ . The syntax to find the Jacobian matrix is

```
Grad[{x,y}, {u,v}]
```

1. Let us consider a simple example where  $x = uv$  and  $y = u + v$ . The Jacobian matrix of the transformation  $(x, y) \rightarrow (u, v)$  and its determinant are computed as follows:

```
x = u*v
y = u+v
J = Grad[{x,y},{u,v}]
Det[J]
```

For a nicer rendition of  $J$ , write

```
MatrixForm[J]
```

2. Sometimes, it might be useful to solve for  $u$  and  $v$  in terms of  $x$  and  $y$ . To do so, try the following:

```
Clear[x,y,u,v]
Solve[{x==u*v, y==u+v}, {u,v}]
```

You will see that there are two possible cases for the pair  $(u, v)$ , but only one of these cases can yield  $u = 1, v = 2, x = 2, y = 3$ . Can you find the Jacobian matrix and determinant of that transformation? The transformation  $(u, v) \rightarrow (x, y)$  is called the *inverse* of the transformation  $(x, y) \rightarrow (u, v)$ .

3. Find the Jacobian matrix and determinant of that transformation

$$x = e^{-r} \cos s, \quad y = e^r \sin s$$

For a 3D change of variable (transformation)  $(x, y, z) \rightarrow (u, v, w)$ , where the old variables are  $(x, y, z)$  and the new variables are  $(u, v, w)$ , the Jacobian matrix is

```
Grad[{x,y,z}, {u,v,w}]
```

4. Find the Jacobian matrix and determinant of the change of variables  $(x, y, z) \rightarrow (u, v, w)$  where  $x = u/v, y = v/w, z = w$ .
5. Find the Jacobian matrix and determinant of the change of variables  $(u, v, w) \rightarrow (x, y, z)$ .

### 3 Sketch the new region after a change of variables

Let  $(x, y)$  be the old variables and  $(u, v)$  be the new variables. Suppose the relation between them are given by  $(u, v) = f(x, y)$ , where  $f$  is a vector function. For example, if  $f(x, y) = (x + y, x)$  then this means  $u = x + y$  and  $v = x$ . The question is: *as  $(x, y)$  varies in a region  $R$ , where does  $(u, v)$  vary in?*

6. Let us consider a specific example where

$$R = \{(x, y) : x^2 + y^2 \leq 1, y \geq 0\}.$$

In other words,  $R$  is the upper half of the unit disk. Let  $u = x + y$  and  $v = x$ . As  $(x, y)$  varies in  $R$ ,  $(u, v)$  varies in some region  $S$  which we want to find.  $S$  is called the *image* of  $R$  under the change of variables (transformation). First, define the region  $R$  as follows:

```
R = ImplicitRegion[x^2+y^2<=1 && y>=0, {x,y}]
```

Then define the change of variables:

```
f[x_,y_] := {x+y, x}
```

Then use the command `TransformedRegion` to get the region  $S$ .

```
S = TransformedRegion[R, f]
```

Then plot  $S$ .

```
Region[S, Axes->True, AspectRatio->Automatic, AxesLabel->{u,v}]
```

7. Draw the image of the region bounded by the parabola  $y = x^2$  and the line  $y = 4$  under the change of variables  $u = x - y, v = x^2 - 2y$ .
8. Draw the image of the unit ball (the interior of the unit sphere) under the change of variables  $u = x, v = x - y, w = z^2$ .

## 4 Visualize vector fields

A *vector field* is a map of arrows: *at each point on the plane or space is placed a vector*. To be more precise, a vector field is a function of two or three variables taking vector values. For example, the gradient of the function  $f(x, y) = x^2 + y^2$ , which is  $\nabla f(x, y) = (2x, 2y)$ , is a vector field because it assigns to each position  $(x, y)$  on the plane a vector  $(2x, 2y)$ . To plot a vector field, use the command **VectorPlot** or **VectorPlot3D**.

9. Try the following:

```
VectorPlot[{2x, 2y}, {x, -2, 2}, {y, -2, 2}]
```

By default, Mathematica scales all the vectors so that they are of the same length while preserving the direction of each vector. The color reflects the actual length. The lighter the color, the longer the vector. To unscale the vectors, add the option **VectorScaling->Automatic** to the previous command. To make all vectors have the same color (blue by default), add the option **VectorColorFunction->None**. To make all vectors red, add the option **VectorStyle->Red**.

10. Plot the vector field  $F(x, y) = (x, x - y)$  in the square  $-1 \leq x, y \leq 1$ .  
11. Plot the vector field  $F(x, y, z) = (yz^2, xz^2, 2xyz)$  with **VectorPlot3D**.

Now think of a vector field as a water current in which the vector at position  $(x, y)$  represents the velocity of the fluid particle at that position. The length (magnitude) of the velocity tells how fast the particle flows. The trajectory of a particle is called a *stream*. Different particles flow along different streams. To visualize the streams, simply replace **VectorPlot** by **StreamPlot**, and **VectorPlot3D** by **StreamPlot3D**.

12. The vector field

$$F(x, y) = \left( \frac{-x^2 + y^2}{(x^2 + y^2)^2} + 1, -\frac{2xy}{(x^2 + y^2)^2} \right)$$

represents a flow passing through a circular object (more precisely, the unit disk).

- (a) Sketch this vector field using **VectorPlot**.  
(b) Sketch the vector field with **StreamPlot** and the unit disk, defined by the inequality  $x^2 + y^2 \leq 1$ , with **RegionPlot** (see Lab 3) on the same plot. To hide the flow inside the circle, make sure to specify the color of **RegionPlot**, such as **PlotStyle->White**.  
13. Visualize the vector field in Problem 11 with **StreamPlot3D**.

## 5 Curl and divergence

You can compute the curl and divergence of a vector field using the commands **Curl** and **Div**, respectively.

14. Consider  $F(x, y, z) = (xyz, xy + z^2, y)$ . Try the following:

```
F[x_,y_,z_] := {x*y*z, x*y+z^2, y}
Curl[F[x,y,z], {x,y,z}]
Div[F[x,y,z], {x,y,z}]
```

15. Let  $F(x, y, z) = (x, y, z)$ ,  $G(x, y, z) = (y^2, xyz, z^2)$ , and  $f(x, y, z) = x/z$ . Find  $\text{curl}(F \times G)$  and  $\text{div}(\text{grad}(f))$ .

## 6 Line integral

Starting from version 13.3 (released in June 2023), Mathematica is equipped with the new command `LineIntegrate`, which significantly simplifies the task of computing line integrals. Recall that there are two types of line integrals:

- scalar line integral:  $\int_C f ds$ , where  $f$  is a scalar-valued function
- vector line integral:  $\int_C \vec{F} \cdot d\vec{r}$ , where  $\vec{F}$  is a vector field.

With `LineIntegrate`, you only need to specify the curve  $C$  and the integrand ( $f$  or  $\vec{F}$ ). Mathematica will take care of the rest.

16. For example, we want to integrate the scalar function  $f(x, y) = x - y$  over upper half of the unit circle. Recall that for scalar line integral, the direction (orientation) of the curve does not matter. This curve is parametrized by  $x = \cos t$  and  $y = \sin t$  where  $0 \leq t \leq \pi$ . To define the curve, we use the command `ParametricRegion` as follows.

```
curve=ParametricRegion[{Cos[t],Sin[t]},{t,0,Pi}]
```

Make sure that you put double curly brackets around `t,0,Pi`. It would be more natural to name the curve 'C' instead of 'curve'. However, 'C' is a protected symbol of Mathematica and is not an available name to use. You can plot this curve to check that Mathematica understands the command correctly:

```
Region[curve]
```

To customize the appearance of the curve, try adding the option `BaseStyle->{Thick,Black}`. Alternatively, you can try

```
Region[Style[curve, Thick, Black]]
```

To compute the line integral, do the following:

```
f[x_,y_] := x-y  
LineIntegrate[f[x,y], {x,y} \[Element] curve]
```

As you type `\[Element]`, it will be turned into the symbol  $\in$ , which is read "is a member of" or "belongs to". You can also type this symbol by typing **ESC el ESC** on your keyboard.

17. Some special curves, such as the the poly-lines, have a built-in structure in Mathematica. A poly-line is a collection of line segments joining a sequence of points. It looks like a zigzag. For example, the command

```
L=Line[{{1,1},{3,1},{1,4},{0,0}}]
```

defines a poly-line connecting (1,1) to (3,1), then from (3,1) to (1,4), and then from (1,4) to (0,0). You can draw  $L$  using the command

```
Region[L]
```

You can change the color of the poly-line as follows

`Region[Style[L,Red]]`

To compute the line integral  $\int_L f(x,y)ds$ , do the following:

`LineIntegrate[f[x,y], {x,y} \[Element] L]`

18. Compute the line integral  $\int_C f(x,y)ds$  where  $C$  is the triangle loop with vertices at  $(2,1)$ ,  $(0,1)$ ,  $(4,3)$  oriented in that order.
19. To compute the vector line integral  $\int_C \vec{F} \cdot d\vec{r}$ , where  $F(x,y) = (y, x^2)$ , do the following:

`F[x_,y_]:= {y,x^2}`

`LineIntegrate[F[x,y], {x,y} \[Element] curve]`

Note that the direction (orientation) of the curve does matter in vector line integrals. The parametrization of the curve lends itself a direction: starting at parameter  $t = 0$ , a point starts from  $(x,y) = (\cos 0, \sin 0) = (1,0)$ , moving along the curve as  $t$  increases, and ends at  $(x,y) = (\cos \pi, \sin \pi) = (-1,0)$ .

Can you plot the vector field  $F$  together with the curve and explain from the picture why result of the line integral should be negative?

20. Recall Green's theorem:

$$\int_C \vec{F} \cdot d\vec{r} = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

where  $\vec{F} = (P, Q)$ ,  $C$  is a simple, closed, positively oriented curve, and  $R$  is the region enclosed by  $C$ . Check this theorem for the case  $F(x,y) = (y, x^2)$  and  $C$  is the triangle with vertices at  $(0,0)$ ,  $(3,1)$ ,  $(1,4)$  oriented in that order. (To integrate over  $R$ , you might want to review Problem 26 of Lab 4).

## 7 Surface integral

Recall that there are two types of surface integrals:

- scalar surface integral:  $\iint_S f dS$ , where  $f$  is a scalar-valued function
- vector surface integral:  $\iint_S \vec{F} \cdot d\vec{S}$ , where  $\vec{F}$  is a vector field.

With the command `SurfaceIntegrate` (available in version 13.3 or higher), you only need to specify the surface  $S$  and the integrand ( $f$  or  $\vec{F}$ ). Mathematica will take care of the rest.

21. You can plot a parametric surface using the command `ParametricRegion` followed by `Region`. For example, the torus parametrized by  $x = (\cos t + 2) \cos s$ ,  $y = (\cos t + 2) \sin s$ ,  $z = \sin t$  with  $t, s \in [0, 2\pi]$  can be drawn as follows.

```
S=ParametricRegion[{(Cos[t]+2)Cos[s], (Cos[t]+2)Sin[s], Sin[t]},  
                  {{t,0,2Pi},{s,0,2Pi}}];  
Region[S]
```

Refer to the [SurfacePlot documentation page](#) to do the following problems.

22. Compute  $\iint_S x^2 dS$  where  $S$  is the torus defined in Problem 21.
23. Compute the flux of the vector field  $F(x,y,z) = (xy, z^2, 1)$  across the torus  $S$ .
24. Compute the flux of the vector field  $F(x,y,z) = (xy, z^2, 1)$  across the tetrahedron with vertices at  $(1,1,0)$ ,  $(1,2,3)$ ,  $(-1,1,0)$ ,  $(3,2,1)$ .