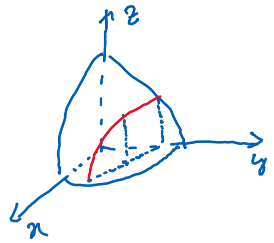
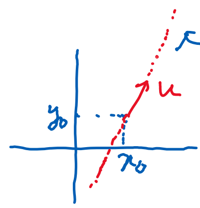
Lecture 17

Friday, May 24, 2024 8:47 AM

* Directional derivative





equation $\begin{cases} x = x_0 + tu_1 \\ y = y_0 + tu_2 \end{cases}$

The red curve is the graph of $g(t) = f(x_0 + tu_1, y_0 + tu_2)$.

The slope of the curve at (x_0, y_0) is $g'(0)$.

By chain rule, $g'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = f_x(x_0 + tu_1, y_0 + tu_2)\,u_1$
$+ f_y(x_0 + tu_1, y_0 + tu_2)\,u_2$.

Substitute $t = 0$:

$$g'(0) = \underbrace{f_x(x_0, y_0)\,u_1 + f_y(x_0, y_0)\,u_2}$$

This is called the directional derivative of f at (x_0, y_0) in the direction of the unit vector u.

Notation:

$$D_u f(x_0, y_0) = f_x(x_0, y_0)\,u_1 + f_y(x_0, y_0)\,u_2$$

$$= \underbrace{\left(f_x(x_0, y_0), f_y(x_0, y_0)\right)}_{\text{gradient vector of } f \text{ at } (x_0, y_0)} \cdot \underbrace{(u_1, u_2)}_{u}$$

$$= \nabla f(x_0, y_0) \cdot u$$

Gradient vector ∇f is equivalent to f' in Calc I.

Ex $f(x,y,z) = x^2y + y^2z + z^2x$

Find directional derivative of f at $(1,2,-1)$ in the direction of the unit vector $u = (\frac{3}{5}, \frac{4}{5}, 0)$.

$$\nabla f = (f_x, f_y, f_z) = (2xy + z^2,\ x^2 + 2yz,\ y^2 + 2zx)$$

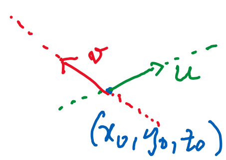
$$\nabla f(1,2,-1) = (5, -3, 2)$$

$$D_u f(1,2,-1) = (5,-3,2) \cdot (\tfrac{3}{5}, \tfrac{4}{5}, 0) = 5(\tfrac{3}{5}) + (-3)\tfrac{4}{5} + 2(0) = \frac{3}{5}.$$

Ex same f as above, same point $(1,2,-1)$.

find the directional derivative in the direction $v = (\frac{1}{3}, \frac{2}{3}, -\frac{2}{3})$.

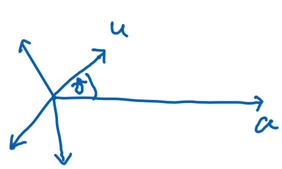
$$D_v f(1,2,-1) = \nabla f(1,2,-1) \cdot v = (5,-3,2) \cdot (\tfrac{1}{3}, \tfrac{2}{3}, -\tfrac{2}{3}) = -\frac{5}{3}$$



f is increasing in the direction u, but decreasing in the direction v.

Question: what is the direction that corresponds to the maximum rate of change ?

$$D_u f(x_0, y_0) = \underbrace{\nabla f(x_0, y_0)}_{a} \cdot u = a \cdot u = |a||u| \cos\theta = |a| \cos\theta$$



$D_u f(x_0, y_0) \to \max$ when $\cos\theta = 1$, which is when $\theta = 0$.

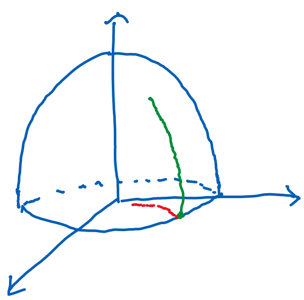
This corresponds to

$$u = \frac{a}{|a|}$$

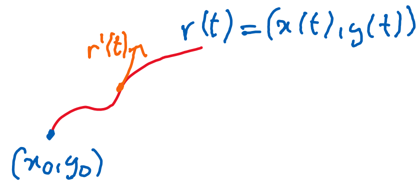
Conclusion: the direction that corresponds to the maximum rate of change of f at (x_0, y_0) is $u = \dfrac{\nabla f(x_0, y_0)}{|\nabla f(x_0, y_0)|}$.

The maximum rate of change is $a \cdot u = a \cdot \dfrac{a}{|a|} = \dfrac{|a|^2}{|a|} = |a| = |\nabla f(x_0, y_0)|$.

$\underline{\underline{Ex}}$ A person wants to climb the mountain $f(x, y) = 9 - x^2 - 2y^2$ from the bottom, at point $(x_0, y_0) = (1, 2)$. Find the path corresponding to the steepest ascend.



The green path on the mountain corresponds to the red path (the projection) on the bottom.



At every point on the red path, the tangent vector $r'(t)$ points to the direction of steepest ascend.

So, $r'(t) = \underset{\substack{\uparrow \\ \text{some positive constant}}}{\lambda} \nabla f$

Thus, $(x', y') = \lambda(-2x, -4y)$

$$\begin{cases} x' = -2\lambda x, & x(0) = 1 \\ y' = -2\lambda y, & y(0) = 1 \end{cases}$$

One can choose $\lambda = 1$ and solve these two ODEs using the separation of variables method.