Lecture 18

Tuesday, May 28, 2024 10:11 AM

* Questions .

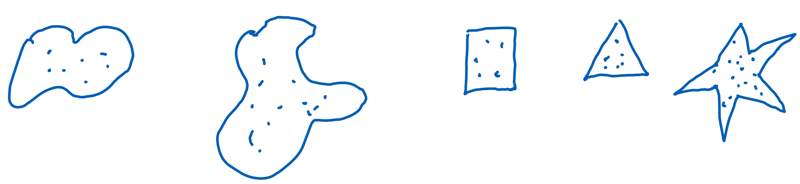
Optimization problem

Calc I: find min/max of $f(x)$ on $[a,b]$

- Find all critical points of f in $[a,b]$, call them $x_1, x_2, x_3, \dots, x_n$
- Compare $f(a), f(x_1), \dots, f(x_n), f(b)$. The largest of those is the maximum. The smallest of those is the minimum.

Multivariable: find min/max of $f(x,y)$ in a region D in $\mathbb{R}^2$.

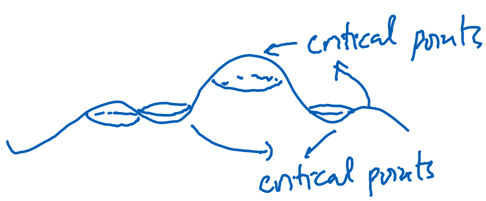
We will assume that D is closed, meaning including its boundary.



The procedure is similar to Calc I:

- Find all the critical points of f in D by solving for (x,y): $\nabla f(x,y) = 0$

  Label the critical points by $(x_1,y_1), (x_2,y_2), (x_3,y_3), \dots, (x_n,y_n)$.



- Compare $f(x_1,y_1), \dots, f(x_n,y_n)$ with f evaluated at the boundary points of D.

There is one issue: there are infinitely many boundary points of D.

Let's take a look at an example.

Ex $f(x,y) = x^3 + xy^2 - 6x + y^2$

$f(x,y) \to$ min/max on the disk of radius $\frac{3}{2}$ centered at the origin.

This is a trickier problem We will learn how to solve it step by step.

Step 1: Find the critical points of the function f

Solve $\nabla f(x,y) = 0$.

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \rightsquigarrow \begin{cases} 3x^2 + y^2 - 6 = 0 \\ 2xy + 2y = 0 \rightsquigarrow 2y(x+1) = 0 \rightsquigarrow y = 0 \text{ or } x = -1 \end{cases}$$

There are two cases: $y = 0$ and $x = -1$.

Case 1: $y = 0$. substitute $y = 0$ into the equation $3x^2 + y^2 - 6 = 0$

$$3x^2 - 6 = 0 \rightsquigarrow x = \pm\sqrt{2}$$

We get two critical points: $(\pm\sqrt{2}, 0)$

Case 2: $x = -1$: substitute $x = -1$ into $3x^2 + y^2 - 6 = 0$

$$3 + y^2 - 6 = 0 \rightsquigarrow y = \pm\sqrt{3}$$

We get two more critical points. $(-1, \pm\sqrt{3})$

In conclusion, there are four critical points: $(\pm\sqrt{2}, 0)$, $(-1, \pm\sqrt{3})$.

Only $(\pm\sqrt{2}, 0)$ lie inside D.

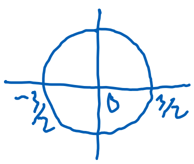
Step 2: $f(\sqrt{2}, 0) = (\sqrt{2})^3 - 6\sqrt{2} = -4\sqrt{2}$

$f(-\sqrt{2}, 0) = 4\sqrt{2}$

We now have to treat points on the boundary of D as a group because we cannot evaluate f at every of them (there are infinitely many such points). We want to find $\min\limits_{\partial D} f$ and $\max\limits_{\partial D} f$ and then compare them with $f(\pm\sqrt{2},0)$. Here ∂D denotes the boundary of D.

On ∂D: $x^2+y^2=\left(\frac{3}{2}\right)^2 \rightsquigarrow y^2 = \frac{9}{4}-x^2$.

$$f(x,y) = x^3 + x\left(\frac{9}{4}-x^2\right) - 6x + \left(\frac{9}{4}-x^2\right) = -x^2 - \frac{15}{4}x + \frac{9}{4}$$



We want to find min/max of $\underbrace{-x^2-\frac{15}{4}x+\frac{9}{4}}_{g(x)}$ on $\left[-\frac{3}{2},\frac{3}{2}\right]$.

$$g'(x) = -2x - \frac{15}{4}$$

$\rightsquigarrow$ critical point of g is $-\frac{15}{8}$, but this point lies outside of $\left[-\frac{3}{2},\frac{3}{2}\right]$.

$$\min_{\left[-\frac{3}{2},\frac{3}{2}\right]} g = \min\left\{g\left(-\tfrac{3}{2}\right), g\left(\tfrac{3}{2}\right)\right\} = g\left(-\tfrac{3}{2}\right) = \frac{45}{8}$$

$$\max_{\left[-\frac{3}{2},\frac{3}{2}\right]} g = \max\left\{g\left(-\tfrac{3}{2}\right), g\left(\tfrac{3}{2}\right)\right\} = g\left(\tfrac{3}{2}\right) = -\frac{45}{8}$$

Therefore,

$$\min_D f = \min\left\{-\frac{45}{8}, \pm 4\sqrt{2}\right\} = -4\sqrt{2}, \text{ attained at } (x,y) = (\sqrt{2},0).$$

$$\max_D f = \max\left\{\frac{45}{8}, \pm 4\sqrt{2}\right\} = 4\sqrt{2}, \text{ attained at } (x,y) = (\sqrt{2},0).$$