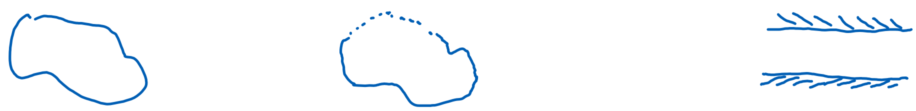
Lecture 19

Wednesday, May 29, 2024 1:29 AM

Last time, we discussed the procedure to find min/max of a function f on a region D. That procedure guarantees that you will be able to find min/max of the function if D is <u>closed</u> and <u>bounded</u>. If D is either not closed or not bounded the min or max might not exist.



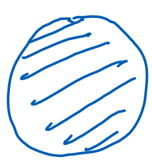
Closed, bounded   bounded, not closed   closed, not bounded

Ex $f(x,y) = x^2+y^2$, $D = \mathbb{R}^2$: closed but not bounded

$\min_D f = 0$

$\max_D f$ DNE

Ex: $f(x,y) = x^2+y^2$, $D = \mathbb{R}^2 \setminus$ unit disk (not including the unit circle)



D

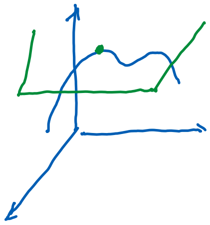
$\left.\begin{array}{l}\min_D f \\ \max_D f\end{array}\right\}$ don't exist

<u>Classification of critical points</u>

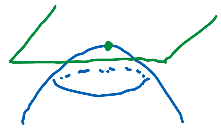
Attention: this has nothing to do with the problem of finding min/max.

(x,y) is called a <u>critical point</u> of f if $\nabla f(x,y) = 0$.

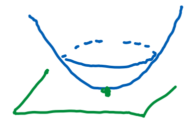
Critical point is where the tangent plane is flat.



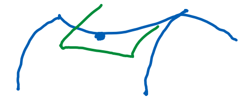
We want to categorize critical points



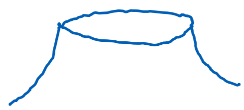
local max



local min



saddle point: min in one path, max in another path



volcano shape

Computationally, how do we tell if a critical point is a local min, local max, or saddle point, or none of those? We need an analogy of the second derivative test in Calc I.

$f(x,y)$ has 4 second derivatives: f_{xx}, f_{xy}, f_{yx}, f_{yy}. A second derivative test will involve all of them.

equal

$$D = \underbrace{\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}}_{\text{Hessian matrix}} = f_{xx} f_{yy} - f_{xy}^2$$

* Second derivative test:

If $D>0$ and $f_{xx}>0$: Local min

If $D>0$ and $f_{yy}<0$: local max

If $D<0$: saddle point

Otherwise: inconclusive

Ex $f(x,y) = x^3 + xy^2 - 6x + y^2$

has 4 critical points : $(\pm\sqrt{2}, 0)$, $(-1, \pm\sqrt{3})$. Classify them.

$$\left.\begin{array}{l} f_x = \\ f_y = \\ f_{xx} = \\ f_{xy} = \\ f_{yy} = \end{array}\right\}$$

At $(\sqrt{2}, 0)$, find D. Is it positive or negative ?

One can point out local min/max, saddle point using contour map.

Contour[f[x,y], {x,-2,2}, {y,-3,3}, Contours→100]

Ex $f(x,y) = 3x - x^3 - 2y^2 + y^4$

Find all the critical points and classify them.