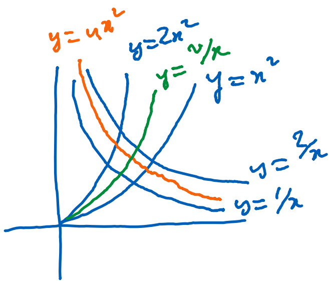
Lecture 31

Thursday, June 13, 2024 1:18 PM

Change of variables:

$$\iint_D f(x,y)\,dA = \iint_{D'} \bar{f}(u,v)\,\overbrace{\underbrace{J\,du\,dv}_{dA'}}^{dA}, \quad \text{where } J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$$

Ex



$$\begin{cases} y = ux^2 \\ y = \frac{v}{x} \end{cases} \qquad 1 \le u \le 2, \quad 1 \le v \le 2$$

$$ux^2 = \frac{v}{x} \rightsquigarrow x = \left(\frac{v}{u}\right)^{1/3} = \frac{v^{1/3}}{u^{1/3}} = u^{-1/3}v^{1/3}$$

$$y = ux^2 = u\frac{v^{2/3}}{u^{2/3}} = u^{1/3}v^{2/3}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \left|\det\begin{pmatrix} -\frac{1}{3}u^{-4/3}v^{1/3} & \frac{1}{3}u^{-1/3}v^{-2/3} \\ \frac{1}{3}u^{-2/3}v^{2/3} & \frac{2}{3}u^{1/3}v^{-1/3} \end{pmatrix}\right| = \left|-\frac{2}{9}u^{-1} - \frac{1}{9}u^{-1}\right| = \frac{1}{3u}$$

$$\iint_D xy\,dA = \iint_{[1,2]\times[1,2]} v\,J\,du\,dv = \int_1^2\int_1^2 v\,\frac{1}{3u}\,du\,dv = \int_1^2 v\frac{1}{3}\ln(u)\Big|_{u=1}^{u=2} dv$$

$$= \frac{1}{3}\ln 2\int_1^2 v\,dv$$

$$= \frac{1}{3}\ln 2\,\frac{v^2}{2}\Big|_1^2 = \frac{\ln 2}{2}$$

* Change of variable for triple integral:

$$\iiint_E f(x,y,z)\,dV = \iiint_{E'} \bar{f}(u,v,w)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

where
$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix} \nabla_{u,v,w}x \\ \nabla_{u,v,w}y \\ \nabla_{u,v,w}z \end{pmatrix} = \det\begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix}$$

For cylindrical coords, $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$

For spherical coords, $\dfrac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \rho^2 \sin\phi$

Vector fields

$F: D \subset \mathbb{R}^2 \to \mathbb{R}^2$ 2D-vector field

$F: E \subset \mathbb{R}^3 \to \mathbb{R}^3$ 3D-vector field

Practice sketching vector fields ....

use Mathematica to plot vector fields ....

* Line integral of scalar functions

$$\int_C f(x,y)\,ds = \int_a^b f(t)\,|r'(t)|\,dt$$

where $r(t)$, $a \le t \le b$, is the parametrization of the curve C.