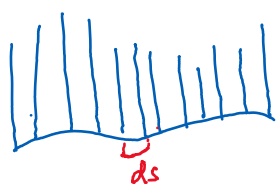
Lecture 33

Tuesday, June 18, 2024 12:53 AM

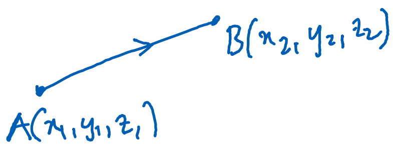
Line integral of scalar functions:

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\,\underbrace{|r'(t)|\,dt}_{ds}$$

infinitesimal length along the curve

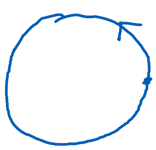


Recall the parametrization of lines and circles:

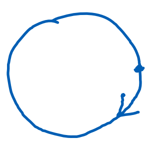


$$\begin{cases} x = (1-t)x_1 + t x_2 \\ y = (1-t)y_1 + t y_2 \\ z = (1-t)z_1 + t z_2 \end{cases} \qquad 0 \le t \le 1$$

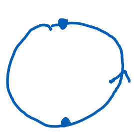
In other words, $r(t) = (1-t)A + tB$



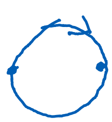
$$\begin{cases} x = r\cos t \\ y = r\sin t \end{cases} \qquad 0 \le t \le 2\pi$$



$$\begin{cases} x = r\cos(2\pi - t) = r\cos t \\ y = r\sin(2\pi - t) = -r\sin t \end{cases} \qquad 0 \le t \le 2\pi$$



$$\begin{cases} x = r\cos t \\ y = r\sin t \end{cases} \qquad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$$



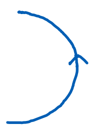
$$\begin{cases} x = r\cos(\pi - t) = -r\cos t \\ y = r\sin(\pi - t) = r\sin t \end{cases} \qquad 0 \le t \le \pi$$

To reverse the direction of a curve $C: r(t),\ a \le t \le b$, we do the following:

$$\bar{C}: r(a+b-t), \quad a \le t \le b$$

Ex Evaluate $$\int_C xy^2\,ds$$

where C is the right half of the unit circle. Note that the direction of C doesn't matter for line integral of scalar functions.



$$\begin{cases} x = \cos t \\ y = \sin t \end{cases} \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$$

$$r'(t) = (x', y') = (-\sin t, \cos t)$$

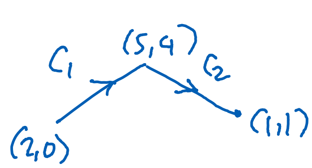
$$|r'(t)| = 1$$

$$\int_C xy^2\,ds = \int_{-\pi/2}^{\pi/2} \cos t \cdot \sin^2 t \cdot 1 \cdot dt = \int_{-\pi/2}^{\pi/2} \cos t \sin^2 t\,dt$$

$$\underset{u=\sin t}{=} \int_{-1}^{1} u^2\,du = \frac{u^3}{3}\Big|_{-1}^{1} = \frac{2}{3}.$$

Ex Evaluate

$$\int_C x\,ds$$ where C is the line segments $(2,0) \to (5,4) \to (1,1)$.



$$C = C_1 \text{ "+" } C_2$$

$$\int_C x\,ds = \int_{C_1} x\,ds + \int_{C_2} x\,ds$$

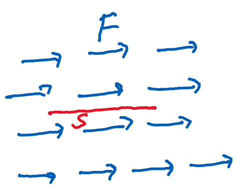
$$C_1: \begin{cases} x = 2(1-t) + 5t = 2+3t \\ y = 0(1-t) + 4t = 4t \end{cases} \qquad C_2: \begin{cases} x = 5(1-t) + 1\cdot t = 5-4t \\ y = 4(1-t) + 1\cdot t = 4-3t \end{cases} \quad 0 \le t \le 1$$

$$\int_{C_1} x\,ds = \int_0^1 (2+3t)\cdot\sqrt{3^2+4^2}\,dt = 5\int_0^1 (2+3t)\,dt = \frac{35}{2}$$

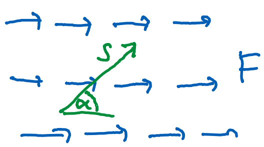
$$\int_{C_2} x\,ds = \cdots = 15$$

$$\int_{C} x\,ds = \frac{35}{2} + 15 = \frac{65}{2}$$

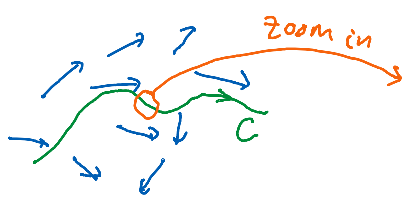
Line integral of a vector field

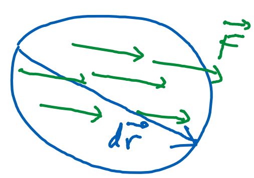


work $= Fs$



work $= F(s\cos\alpha) = \vec{F}\cdot\vec{s}$





work on $d\vec{r}$ is $\vec{F}\cdot d\vec{r}$

Total work is $\int_C \vec{F}\cdot d\vec{r}$