Lecture 34

Thursday, June 20, 2024 8:28 AM

Line integral of a scalar function:

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\underbrace{|r'(t)|dt}_{ds}$$

Line integral of a vector function:

$$\int_C \vec{F}(x,y)\cdot d\vec{r} = \int_a^b \vec{F}(x,y)\cdot \vec{r}'(t)\,dt$$

We will drop the vector symbol on $\vec{F}$ and $\vec{r}$ for simplicity.

$$r(t) = (x(t), y(t))$$

$$r'(t) = (x'(t), y'(t))$$

$$dr = r'dt = (x', y')dt = (x'dt, y'dt) = (dx, dy)$$

$$F = (P, Q)$$

So, $\int_C F\cdot dr = \int_C (P,Q)\cdot(dx,dy) = \int_C P\,dx + Q\,dy \longleftarrow$ The textbook uses this notation.

$\int_C F\cdot dr = \int_a^b (Px' + Qy')\,dt$ = the total work done by the force field F as a particle moves along C.

This work is positive if the force field helps the motion, negative if it hinders the motion.

Ex Let $F(x,y) = (y, x^2)$

Let C be the line segment starting at $(0,-2)$ and ending at $(2,2)$.

Compute $\int_C F \cdot dr$

Note: another of writing this integral is $\int_C y\,dx + x^2 dy$

Use Mathematica to sketch the curve inside the vector field and evaluate the line integral.

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curve = ParametricRegion[{2 t, -2 + 4 t}, {{t, 0, 1}}]
F[x_, y_] := {y, x^2}
p1 = Region[curve];
p2 = VectorPlot[F[x, y], {x, -2, 2}, {y, -2, 2}];
Show[p1, p2]
LineIntegrate[F[x, y], {x, y} \[Element] curve]
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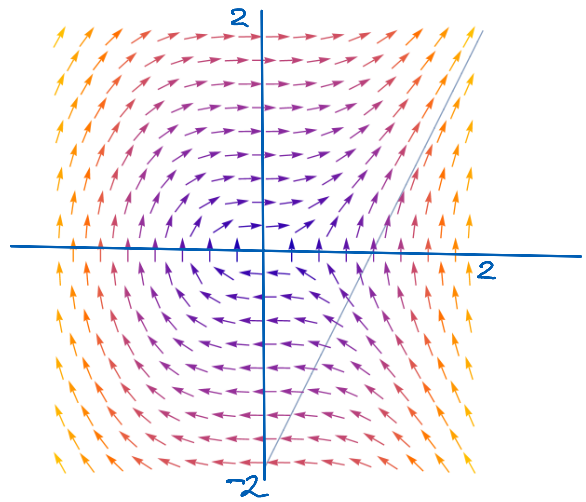

$$\begin{cases} x = 0 + 2t \\ y = -2 + 4t \end{cases} \quad 0 \le t \le 1$$

$$r = (x, y) = (2t, -2+4t)$$

$$r' = (x', y') = (2, 4)$$

$$F = (y, x^2) = (-2+4t, 4t^2)$$

$$F \cdot r' = (-2+4t, 4t^2) \cdot (2, 4) = 16t^2 + 8t - 4$$

$$\int_C F \cdot dr = \int_0^1 (16t^2 + 8t - 4)\,dt = \frac{16}{3}$$

Ex Do the same, but now C is the half circle center at the origin starting at $(2,0)$ and ending at $(-2,0)$.

$$C: \begin{cases} x = 2\cos t \\ y = 2\sin t \end{cases} \quad 0 \le t \le \pi$$