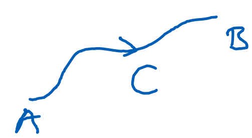
Lecture 35

Thursday, June 20, 2024 11:15 AM

* Fundamental theorem of Calculus for Multivariable functions:

$$\int_C F \cdot dr = \phi(B) - \phi(A)$$



if $F = \nabla \phi$ for some scalar function ϕ.

If $F = \nabla\phi$ for some scalar function ϕ, F is called a potential vector field.

Ex the gravitional field is potential

2D: $F(x,y) = -\alpha \dfrac{(x,y)}{|(x,y)|^3} = -\alpha \nabla\left(\dfrac{1}{\sqrt{x^2+y^2}}\right)$

3D: $F(x,y) = -\alpha \dfrac{(x,y,z)}{|(x,y,z)|^3} = -\alpha \nabla\left(\dfrac{1}{\sqrt{x^2+y^2+z^2}}\right)$

Ex $F(x,y) = (y,-x)$ is not potential.

Suppose it is: $F = \nabla\phi$

$\rightsquigarrow \phi_x = y, \ \phi_y = -x$

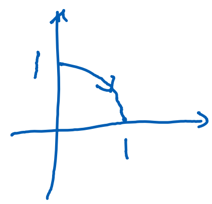
$\rightsquigarrow \phi_{xy} = 1, \ \phi_{xy} = -1$ (impossible)

Thm $F(x,y) = (P(x,y), Q(x,y))$

is a potential vector field if $P_y = Q_x$.

Ex Find $\int_C F\cdot dr$

where $F = (e^x + y, 2y + x)$ and C is the unit circle from $(0,1)$ to $(1,0)$ clockwise.



$P = e^x + y$

$Q = 2y + x$

$P_y = 1, \ Q_x = 1 \rightsquigarrow F$ is a potential vector field

Find potential function: $F = \nabla\phi$

$$\rightsquigarrow \begin{cases} \phi_x = P = e^x + y \rightsquigarrow \phi = e^x + xy + C(y) \rightsquigarrow \phi_y = x + C'(y) \\ \phi_y = Q = 2y + x \end{cases}$$

$$\rightsquigarrow C'(y) = 2y \rightarrow C(y) = y^2$$

Thus, $\phi(x,y) = e^x + xy + y^2$

$$\int_C F\cdot dr = \phi(1,0) - \phi(0,1) = e - (1+1) = e - 2.$$