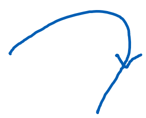
Lecture 36

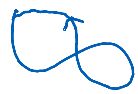
Monday, June 24, 2024 9:30 AM

Green's theorem

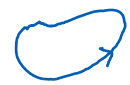
$\int_C F\cdot dr$ where C is a simple closed curve



not closed



closed, not simple

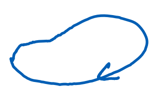


closed and simple

$$\int_C F\cdot dr = \pm \iint_D (Q_x - P_y)\, dA$$

The sign is determined by the orientation of D.

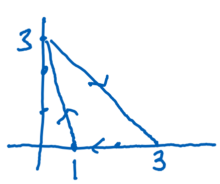
positive orientation



negative orientation

Ex Evaluate $\int_C F\cdot dr$

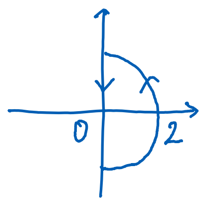
where C is the triangle $(0,3)\to(3,0)\to(1,0)\to(0,3)$ and $F(x,y)=(y^2, x-y)$.



Ex

$$\int_C F\cdot dr$$

where C is as in the picture and $F=(-y^3+x,\ x^3+y)$.

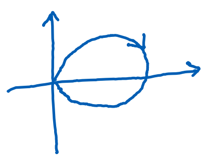


$$\int_C F\cdot dr = \iint_D (Q_x - P_y)\,dA = \iint_D (3x^2+3y^2)\,dA = 3\iint_D (x^2+y^2)\,dA$$

$$= 3\int_0^2 \int_{-\pi/2}^{\pi/2} r^2\, r\, d\theta\, dr = 3\int_0^2 \pi r^3\, dr = 3\pi \frac{r^4}{4}\Big|_0^2$$

$$= 12\pi$$

Ex Find the area enclosed by the curve $r(t) = (\sin t, \sin 2t)$, $0 \le t \le \pi$.



$$\iint_D 1\, dA = \iint_D Q_x - P_y\, dA \quad \text{, where } Q = x,\ P = 0$$

$$= -\int_C F\cdot dr \qquad \text{where } F = (P, Q) = (0, x)$$

$$= -\int_C x\, dy$$

$$= -\int_0^\pi x y'\, dt = -\int_0^\pi \sin t\ 2\cos 2t\, dt = -2\int_0^\pi \sin t \cos 2t\, dt$$

Let $u = \cos t \rightsquigarrow du = -\sin t\, dt$

$$\text{area} = 2\int_1^{-1} (2u^2-1)\,du = -2\int_0^1 (2u^2-1)\,du = -2\left(\frac{2u^3}{3} - u\right)\Big|_0^1 = \frac{2}{3}.$$

* Curl of a vector field:

$$\text{curl } F = \nabla \times F, \quad \text{where } \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \text{ or } \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

Some books (especially in physics) use "rot" instead of "curl".

If $F = (P, Q)$ (2D vector field):

$$\text{curl} F = \nabla \times F = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \times (P, Q) = \begin{vmatrix} \partial/\partial x & \partial/\partial y \\ P & Q \end{vmatrix} = Q_x - P_y$$

If $F = (P, Q, R)$ (3D vector field)

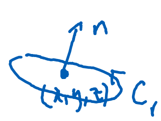
$$\text{curl} F = \nabla \times F = \left(R_y - Q_z, P_z - R_x, Q_x - P_y\right)$$

$$\begin{array}{ccccc} \partial/\partial x & \partial/\partial y & \partial/\partial z & \partial/\partial x & \partial/\partial y \\ P & Q & R & P & Q \end{array}$$

C_r = circle of radius r, positively oriented

$$\frac{1}{\pi r^2} \int_{C_r} F \cdot dr = \text{average circulation density on the disk with radius } r.$$

$$\text{curl} F = \lim_{r \to 0} \frac{1}{\pi r^2} \int_{C_r} F \cdot dr \; : \quad \text{circulation density at } (x, y)$$



In 3D, $(\text{curl} F) \cdot n = \lim_{r \to 0} \frac{1}{\pi r^2} \int_{C_r} F \cdot dr$

circulation density at (x, y, z) in the direction n.

Ex $F = (y, 0)$

$F = (y, -x)$

$F = \left(\frac{y}{\sqrt{x^2 + y^2}}, \frac{-x}{\sqrt{x^2 + y^2}}\right)$

* Divergence of a vector field

$$\int_{C_r} F\cdot n\, ds = \int_{C_r} F\cdot dn \ : \text{ divergence across } C_r$$

$$\frac{1}{\pi r^2}\int_{C_r} F\cdot dn \ : \text{ average divergence on the disk of radius } r$$

$$\operatorname{div} F(x,y) = \lim_{r\to 0} \frac{1}{\pi r^2}\int_{C_r} F\cdot dn$$

$$\operatorname{div} F(x,y,z) = \lim_{r\to 0} \frac{1}{\frac{4}{3}\pi r^3} \underbrace{\int_{S_r} F\cdot dS}_{\text{surface integral}}$$

$$\operatorname{div} F = \nabla\cdot F$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\cdot (P,Q) = P_x + Q_y$$

Ex

$F = (x,y)$

$F = (xy, x)$